An application of almost invariant sets and uniqueness property of invariant measures

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Abstract. It is shown that $\mathbf{R}^\omega$ can be represented us the union of two disjoint almost invariant sets.

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The main purpose of this paper is to consider some properties of almost invariant sets and their applications in the infinite-dimensional topological vector space.

Throughout this article, we use the following standard notation:
$\mathbf{R}$ is the set of all real numbers;
ω is the first infinite cardinal number;
$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^\omega$);
$\mathrm{dom}(\mu)$ is the domain of a given measure μ;
μ' is the completion of a given measure μ;
$\mathbf{R}^\omega$ is the space of all real-valued sequences;
$B(\mathbf{R}^\omega)$ is the σ-algebra of all Borel subsets in $\mathbf{R}^\omega$.
Let E be a nonempty set, G be a group of transformations of E.
Let X be a subset of E. We say that X is almost G-invariant if, for each transformation $g \in G$ we have the inequality

$$card(g(X) \triangle X) < card(E),$$

where the symbol $\triangle$ denotes, as usual, the operation of the symmetric difference of two sets.

From the above definition of the almost G-invariant set, the following two assertions are true:

1. if a set X is almost G-invariant, then the set $E \setminus X$ is almost G-invariant, too;
2. if the sets X and Y are almost G-invariant, then the set $X \cup Y$ is almost G-invariant, too;

In particular, the family of all almost G-invariant sets forms an algebra of subsets of E.

Property of an almost invariant sets is frequently crucial in the process of investigation of many significant questions of the theory of invariant and quasi-invariant measures. For instance, some applications of such sets to the theory of invariant extensions of the Lebesgue measure on the n-dimensional Euclidean space $\mathbf{R}^n$ are considered in the works [1], [2]. Namely, in the paper by Kakutani and Oxtoby [1], a certain method was developed, by means of which it is possible to construct a nonseparable $\mathbf{R}^n$-invariant extension of the lebesgue measure.This method is essentially based on some deep properties of almost invariant subsets of $\mathbf{R}^n$.

It is known that in infinite-dimensional vector spaces there are no analogies of the classical Lebesgue measure. In other words, the above-mentioned spaces do not admit nontrivial, σ-finite translation-invariant Borel measure. In this context notice that A. Kharazishvili constructed a normalized σ-finite metrically transitive Borel measure χ in $\mathbf{R}^\omega$, which is invariant with respect to the everywhere dense vector subspace G of $\mathbf{R}^\omega$, where

$$G = \{x : x \in \mathbf{R}^\omega, card\{i : i \in \omega : x_i \neq 0\} < \omega\}.$$

We put

$$A_n = \mathbf{R}_1 \times \mathbf{R}_2 \times \cdots \times \mathbf{R}_n \times (\prod_{i>n} \triangle_i),$$

where $n \in \mathbf{N}$ and

$$(\forall i)(i \in \mathbf{N} \Rightarrow \mathbf{R}_i = \mathbf{R} \wedge \triangle_i = [0, 1]).$$

For arbitrary natural number $i \in \mathbf{N}$, consider the Lebesgue measure μ_i defined on the space $\mathbf{R}_i$ and satisfying the condition $\mu_i(\triangle_i) = 1$. Let us denote by λ_i the normed Lebesgue measure defined on the $\triangle_i$. In other words, $\lambda_i(\triangle_i) = 1$.

For arbitrary $n \in \mathbf{N}$, let us denote by χ_n the measure defined by

$$\chi_n = (\prod_{1 \le i \le n} \mu_i) \times (\prod_{i>n} \lambda_i),$$

and by $\overline{\chi_n}$ the Borel measure In the space $\mathbf{R}^\omega$ defined by

$$\overline{\chi_n} = \chi_n(X \cap A_n), \qquad X \in B(\mathbf{R}^\omega).$$

Lemma. For arbitrary Borel set $X \in B(\mathbf{R}^\omega)$ there exists a limit

$$\chi(X) = \lim_{n \to \infty} \overline{\chi_n}(X).$$

Moreover, the functional χ is a nonzero σ-finite measure on the Borel σ-algebra $B(\mathbf{R}^\omega)$, which is invariant with respect to the group generated by the everywhere dense vector subspace G and the central symmetry of $\mathbf{R}^\omega$.

Let χ' denotes the completion of measure χ. In other words, χ' is the complete G-measure in $\mathbf{R}^\omega$.

Let s_0 be the central symmetry of $\mathbf{R}^\omega$ with respect to the origin;

S_ω be the group generated by s_0 and G.

It is not hard to verify that the linear hull (over Q) of the set $\{e_\xi : \xi < \alpha\}$ coincide with $\mathbf{R}^\omega$, where $\{e_\xi : \xi < \alpha\}$ is the Hamel basis in $\mathbf{R}^\omega$.

For any $x \in \mathbf{R}^\omega$, we have a unique representation

$$x = \sum_{\xi<\alpha} q_\xi e_\xi,$$

where all $q_\xi \quad (\xi < \alpha)$ are rational numbers and

$$card(\{\xi < \alpha : q_\xi \neq 0\}) < \omega.$$

For each $x \in \mathbf{R}^\omega \setminus \{0\}$ denote by $\xi(x)$ the largest ordinal from the interval $[0, \alpha)$ satisfying the relation $q_{\xi(x)} \neq 0$ and define

$$A = \{x \in \mathbf{R}^\omega : q_{\xi(x)} > 0\},$$

$$B = \{x \in \mathbf{R}^\omega : q_{\xi(x)} < 0\}.$$

It is clear that

$$\mathbf{R}^\omega = A \cup B \cup \{0\}.$$

The following statement is valid.

Theorem 1. There exists a partition $\{A, B\}$ of $\mathbf{R}^\omega$ satisfying next three conditions:

(1)$(\forall F)(F \subset \mathbf{R}^\omega$, F is a closed subset, $\chi'(F) > 0 \Rightarrow card(A \cap F) = card(B \cap F) = \mathbf{c})$;

(2)$(\forall g)(g \in G \Rightarrow card(A \triangle g(A)) < \mathbf{c}, card(B \triangle g(B)) < \mathbf{c})$;

(3)$(\forall h)(h \in s_0 \Rightarrow h(B) = A \cup \{0\}$, where $\{0\}$ is the neutral element of additive group $\mathbf{R}^\omega)$.

Analogous partitions of n-dimensional Euclidean spaces can be found in the works [3], [4], [5].

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