Shalva Pkhakadze

Some Questions of Notation Theory



$$\exists_1 \mathsf{x}\, \mathbb{A} \text{ ——— } \exists \mathsf{x}\, \mathbb{A} \wedge \forall \mathsf{x}\, \forall \mathsf{y}\, [\mathbb{A} \wedge (\mathsf{y}/\mathsf{x})\mathbb{A} \longrightarrow \mathsf{x} = \mathsf{y}]$$

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Chapter 1

Introduction

The development of formal mathematics has led to significant discoveries—significant results—that relate not only to the foundations of mathematics but also to its special branches. At present, obtaining such results without resorting to the methods of formal mathematics is considered hopeless. Therefore, mastering the methods of formal mathematics is very important for the development of all of mathematics. Nevertheless, most areas of mathematics are at present developed at the informal, contentual level. This is due to the fact that mastering the methods of formal mathematics is difficult, because—and this is the main point—there are as yet no perfect expositions of formal mathematical systems. To improve the existing expositions, a number of deep scientific investigations are required.

At present a number of systems of formal mathematical theories are in use, having important applications in various areas of science. Research continues with the aim of creating the most perfect formal mathematical system. The creation of an optimal and convenient formal mathematical system, and the transfer into it of the results obtained in the existing formal systems, is an important and difficult problem.

The present work investigates a part of this problem. Namely, we introduce and investigate rules for the definition of contracting symbols.

Usually the rules for defining contracting symbols impose no restrictions, or very few, so that it is impossible to establish such general properties of contracted forms as would give sufficient freedom in operating with contracted forms. Therefore, instead of operating directly on contracted forms, one adheres to the principle according to which a contracted form is regarded as the form denoted by it. But the full implementation of this principle is impossible, because the forms denoted by contracted forms are almost always very long. Hence it is very important to have the possibility of operating directly on contracted forms. For this we need general laws of operating on contracted forms, as well as laws establishing that actions on contracted forms correspond to analogous actions on the corresponding forms. This, in its turn, requires the possibility of establishing general properties of contracting symbols and contracted forms.

The introduction of derived operators (contracting symbols) is sometimes accompanied by a transition to an extended theory. This also causes great difficulties.

In this work, particular attention is paid to the creation of a general and conve-

nient system of rules for the definition of contracting symbols. A system of rules of this kind must be so general as to satisfy fully the needs of all the known basic formal mathematical theories; at the same time, the contracted forms obtained by applying these rules must have the best possible properties. In a certain sense these two requirements contradict each other. This is why it is hard to find an optimal system of definition rules. The systems of rules for the definition of contracting symbols presented in the monographs [5] and [7] come closer to complete rigor than the others, but they do not have the required generality.

The creation of an optimal system of rules for the definition of contracting symbols is of great theoretical and practical importance. For example, on the basis of such rules the writing of monographs and textbooks and their mastery become easier.

In this work we find a reasonably optimal system of rules for the definition of contracting symbols in the case of classical formal mathematical theories. This system is optimal in the following sense. On the one hand, it is so general that with its help one can introduce essentially all the contracting symbols used in classical formal mathematical theories; on the other hand, it has such rich properties that there is sufficient freedom in operating on contracted forms.

Namely, we introduce the types I–VII, II′, IV′, VI′ of rules for the definition of contracting symbols. The system of rules corresponding to the types I–IV, II′ and IV′ is optimal in the above sense. A consideration of the types V–VII and VI′ shows that the slightest weakening of the requirements given in the definition of the types I–IV, II′ and IV′ is enough for many important properties of contracted forms to cease to hold.

In addition, we study the algorithmic processes of reconstructing forms from a contracted form. We find an algorithm for reconstructing a form from a contracted form with the minimal number of steps in a certain (sufficiently wide and naturally introduced) class of algorithms. We also study various numerical characteristics of contracting symbols and contracted forms, and find algorithms for computing these numerical characteristics.

Chapter 2

Basic Definitions

2.1 Definition of a General Formal Theory $\mathscr{T}$

We shall have in mind a sufficiently general formal mathematical theory $\mathscr{T}$. We assume that the alphabet of the theory $\mathscr{T}$ contains all or some of the symbols listed below.

1. **Propositional variables**

$$\mathbb{P}_0,\ \mathbb{P}_1,\ \mathbb{P}_2,\ \ldots \tag{1}$$

and the **basic proper propositional constants** t and f, and the **basic improper propositional constants**

$$\mathbb{P}'_0,\ \mathbb{P}'_1,\ \mathbb{P}'_2,\ \ldots \tag{2}$$

In addition to these last, there may be, in finite or countable number, other basic propositional constants. The term **propositional letter** is, by definition, the generic name for propositional variables and basic propositional constants.

2. **Object variables**

$$\mathbb{S}_0,\ \mathbb{S}_1,\ \mathbb{S}_2,\ \ldots \tag{3}$$

and the **basic improper object constants**

$$\mathbb{S}'_0,\ \mathbb{S}'_1,\ \mathbb{S}'_2,\ \ldots \tag{4}$$

In addition to these last, there may be, in finite or countable number, other basic proper or improper object constants. The term **object letter** is, by definition, the generic name for object variables and basic object constants.

3. **Basic simple logical operators** (in finite or countable number). Each of them is either **relational** or **substantive**. The basic simple relational logical operators are also called **basic logical connectives** or **basic propositional connectives**.

4. n**-place predicate variables**, or n**-place relational functional variables**

$$\mathbb{P}^n_0,\ \mathbb{P}^n_1,\ \mathbb{P}^n_2,\ \ldots, \tag{5}$$

$n = 1, 2, \ldots$, and the **basic improper n-place predicate constants**, or **basic n-place relational functional constants**

$$\overline{\mathbb{P}}^n_0,\ \overline{\mathbb{P}}^n_1,\ \overline{\mathbb{P}}^n_2,\ \ldots, \tag{6}$$

$n = 1, 2, \ldots$ In addition to these last, there may be, in finite or countable number, other basic proper or improper n-place predicate constants. As the generic name for n-place predicate variables and basic n-place predicate constants, we use the term **n-place predicate letter**, or **n-place relational functional letter**, or **basic simple n-place relational special operator**, or **basic n-place relational functional operator**, or **basic n-place predicate**.

5. **n-place substantive functional variables**

$$\varphi^n_0,\ \varphi^n_1,\ \varphi^n_2,\ \ldots, \tag{7}$$

$n = 1, 2, \ldots$, and the **basic improper n-place substantive functional constants**

$$\overline{\varphi}^n_0,\ \overline{\varphi}^n_1,\ \overline{\varphi}^n_2,\ \ldots, \tag{8}$$

$n = 1, 2, \ldots$ In addition to these last, there may be, in finite or countable number, other basic proper or improper n-place substantive functional constants. As the generic name for n-place substantive functional variables and basic n-place substantive functional constants, we use the term **n-place substantive functional letter**, or **basic simple n-place substantive special operator**, or **basic n-place substantive functional operator**, or **basic n-place function**.

6. **Basic simple logical-special operators** (in finite or countable number). Each of them is either **relational** or **substantive**.

7. **Basic operator signs** (in finite or countable number). Each of them is: (a) either **logical**, or **special**, or **logical-special**; (b) either **relational** or **substantive**; (c) either **total** or **partial**.

It is required that the alphabet of the theory $\mathscr{T}$ satisfy at least one of the following two conditions:

(a) the alphabet of $\mathscr{T}$ contains propositional variables (1) or propositional constants (2), and the class of operators of $\mathscr{T}$ (the class of operators of $\mathscr{T}$ is defined below) contains at least one (non-nullary) logical operator;

(b) the alphabet of $\mathscr{T}$ contains object variables (3) or object constants (4), and the class of operators of $\mathscr{T}$ contains at least one (non-nullary) special operator.

Moreover, the following two conditions are required to hold:

(c) if the alphabet of $\mathscr{T}$ contains at least one member of some sequence among (1)–(8), then every member of this sequence is contained in the alphabet of $\mathscr{T}$;

(d) if the class of operators of $\mathscr{T}$ contains at least one non-nullary special operator, then condition (b) holds.

As the generic name for the various types of variables, resp. basic constants, resp. symbols, of the theory $\mathscr{T}$ we use the term **variable**, resp. **basic constant**, resp. **basic symbol**.

The order of symbols determined by some of the sequences among (1)–(8) is called the **alphabetic order** of the corresponding symbols. These sequences are called the **alphabetic enumerations** of the corresponding symbols. To clarify the meaning of this definition, note, for example, that if the alphabet of the theory $\mathscr{T}$ contains the sequences (1) and (2), then the alphabetic enumeration of propositional letters is to be regarded as the transfinite sequence

$$\mathbb{P}_0,\ \mathbb{P}_1,\ \mathbb{P}_2,\ \ldots;\ \mathbb{P}'_0,\ \mathbb{P}'_1,\ \mathbb{P}'_2,\ \ldots\ .$$

Accordingly, if of two propositional letters p and q one is a variable and the other a constant, then among p and q we regard the variable as the letter with the **minimal index** (in other cases the meaning of the latter term is understood as usual).

Thus we have letters, resp. variables, resp. basic constants, of the following (basic) types: propositional, object, n-place substantive special for $n = 1, 2, \ldots$, and n-place relational special for $n = 1, 2, \ldots$.

To every basic operator sign of the theory $\mathscr{T}$ there is assigned a pair of non-zero natural numbers, called the **weight** of this operator sign. If the weight of a basic operator sign σ is (n, m), then all of the words of the form $\sigma\,\mathsf{a}_1 \cdots \mathsf{a}_m$, where $\mathsf{a}_1, \ldots, \mathsf{a}_m$ are letters from the sequences (1)–(8), or, according to some special definition, some of them, are called **basic compound operators** or **basic quantifiers**. Depending on what kind of operator sign σ is, these operators are called, respectively, **logical**, **special**, **relational**, **substantive**, **logical-special**, **partial**, **total**. To every basic operator there is assigned, as its **weight**, a non-zero natural number, in such a way that the weight of an operator obtained from an operator sign of weight (n, m) is n, and the weight of an operator among those in the sequences (5)–(8) is its upper index. An operator of weight n is called **n-place**, and an operator sign of weight (n, m) is called **n-place m-ary**. Instead of the terms 1-ary, 2-ary, 3-ary, we also use the terms **singulary** or **unary**, **binary** and **ternary**. We see that every basic operator is either simple or compound. In compound operators of the form $\sigma\,\mathsf{a}_1 \cdots \mathsf{a}_m$ the letters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ (we mean occurrences of letters) are called **operator letters**, and a_k is called the **k-th operator letter** ($k = 1, 2, \ldots, m$). The letters occurring in the alphabetic enumeration of the letters of the given theory we call **quantifier letters** (in contrast to operator letters, quantifier letters are letters, not occurrences of letters). In connection with this, the terms **operator variable**, **operator constant**, **quantifier variable** and **quantifier constant** are used in the obvious sense. A quantifier letter a may fail to be an operator letter of any quantifier, but there exist theories of the type $\mathscr{T}$, obtained as extensions of the theory $\mathscr{T}$, in which there are quantifiers with the operator letter a. Moreover, only quantifier letters have this property.

To every basic n-place logical-special operator sign, resp. operator, there is as-

signed, besides its weight, a **logicality indicator** in the form of a non-empty ordered system of natural numbers $(n_1, \ldots, n_k)$, $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. As the logicality indicator of an operator (quantifier) obtained by applying an n-place logical-special operator sign σ, one should take the logicality indicator of the operator sign σ. Further, to every basic n-place partial operator sign σ there is assigned a **binding indicator** in the form of a non-empty ordered system of natural numbers $(n_1, \ldots, n_k)$, $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. The same system is assigned as the binding indicator to every compound operator (quantifier) obtained by applying the operator sign σ. We shall assume (and this is natural) that the logicality indicator of any n-place logical, resp. special, operator sign or operator is $(1, 2, \ldots, n)$, resp. $(\,)$ (by $(\,)$ we denote the empty ordered system, which is the empty set). Similarly, we shall assume that the binding indicator of any n-place total operator sign or quantifier is $(1, 2, \ldots, n)$.

The class of basic operators of the theory $\mathscr{T}$ is required to satisfy the following conditions: (1) if in one of two compound operators obtained by applying the basic operator sign σ we replace the k-th operator letter by the k-th operator letter of the other operator, then we obtain a (compound) operator; (2) if in a compound operator we replace an operator constant by a quantifier letter of the same type (the latter may also be a variable), then we obtain an operator; (3) if in a compound operator we replace an operator variable by a variable of the same type, then we obtain an operator.

Basic propositional constants are also called **basic nullary relational logical operators** and, consequently, they are **basic nullary propositional (logical) connectives**. Propositional letters are also called **basic nullary relational special (functional) operators** or **basic nullary predicates**, and object letters are called **basic nullary substantive special (functional) operators**.

Below, wherever we speak of basic operators and wish to include basic nullary operators as well, we shall use clarifying phrases such as "including operators of weight 0", "including nullary logical operators", etc., or at least it must follow clearly from the context that basic nullary operators (of the appropriate type) are also meant.

The notions of **formula** and **term** in the theory $\mathscr{T}$ are defined as follows. Propositional letters are (by definition) (the **simplest**) formulas, and object letters are (the **simplest**) terms. If σ is a basic n-place logical, resp. special, resp. logical-special operator with logicality indicator $(n_1, \ldots, n_k)$, and $\mathbb{A}_1, \ldots, \mathbb{A}_n$ are formulas, resp. terms, resp. such forms that $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_k}$ is a subsequence of maximal length of the sequence $\mathbb{A}_1, \ldots, \mathbb{A}_n$ each member of which is a formula, then $\sigma\mathbb{A}_1 \cdots \mathbb{A}_n$ is either a formula or a term, according as the operator σ is relational or substantive. Here, as well as below, in recursive definitions we, as a rule, give only the direct clauses of the definition.

As the generic name for formulas and terms we use the term **form**. Formulas are called **propositional forms**, and terms are called **object forms**. Moreover, we shall say that formulas, resp. terms, are **forms of one and the same type**. The simplest formulas, and the formulas of the form $\sigma\,\mathbb{X}_1 \cdots \mathbb{X}_n$, where σ is a basic n-place simple special relational operator and $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are object letters, are called **atomic formulas**.

We shall not fix axioms and rules of inference. The notions of proof and theorem are introduced as usual.

2.2 Theories $\mathscr{T}'$ and $\mathscr{T}''$. Examples

2.2.1 Theories $\mathscr{T}'$ and $\mathscr{T}''$

Together with the theory $\mathscr{T}$ we shall consider the theories $\mathscr{T}'$ and $\mathscr{T}''$, which are in a certain sense equivalent to the theory $\mathscr{T}$. They are defined as follows. The alphabets of the theories $\mathscr{T}'$ and $\mathscr{T}''$ coincide, and their common alphabet is obtained from the alphabet of the theory $\mathscr{T}$ by adding the symbols "[" and "]". To obtain the definitions of the notions of formula and term in $\mathscr{T}''$, it suffices to replace, in the definition of these notions in $\mathscr{T}$, the expression $\sigma\mathbb{A}_1 \cdots \mathbb{A}_n$ by the expression $[\sigma\mathbb{A}_1 \cdots \mathbb{A}_n]$. To obtain the definitions of these notions in $\mathscr{T}'$, one has to declare some basic two-place operators to be **labeled operators** and to replace, in the definition of the notions of formula and term in $\mathscr{T}$, the phrase "$\sigma\mathbb{A}_1 \cdots \mathbb{A}_n$ is either a formula or a term" by the phrase "either $[\mathbb{A}_1\, \sigma\, \mathbb{A}_2]$ or $[\sigma\mathbb{A}_1 \cdots \mathbb{A}_n]$, according as σ is a labeled operator or not, is either a formula or a term". We shall always assume that the labeled operators are the two-place simple operators that are not quantifier letters, and only they.

Natural one-to-one correspondences are established between the forms of the theories $\mathscr{T}$, $\mathscr{T}'$ and $\mathscr{T}''$. Moreover, if $\mathbb{A}$, $\mathbb{A}'$ and $\mathbb{A}''$ are corresponding forms of the theories $\mathscr{T}$, $\mathscr{T}'$ and $\mathscr{T}''$, then natural one-to-one correspondences are established between the parts of the forms $\mathbb{A}$, $\mathbb{A}'$ and $\mathbb{A}''$. It is not difficult to formulate explicitly the algorithms that establish these one-to-one correspondences. For example, to obtain $\mathbb{A}$ from $\mathbb{A}''$ it suffices to delete all the brackets; and, accordingly, to obtain $\mathbb{A}''$ from $\mathbb{A}$ it suffices to regard $\mathbb{A}$ as the "simplified form" of a form of the theory $\mathscr{T}''$, obtained by deleting all the brackets, and to reconstruct the brackets (the result of such a reconstruction is, as is easily seen, determined uniquely). We shall assume that the deletion, resp. the reconstruction, of brackets is carried out step by step, in such a way that every step means the deletion, resp. the reconstruction, of a single pair of brackets. The intermediate expressions obtained in these processes we call **intermediate forms** (**intermediate formulas** and **intermediate terms**) **with respect to $\mathscr{T}$ and $\mathscr{T}''$**. Further, it is easy to see that in order to obtain $\mathbb{A}'$ from $\mathbb{A}''$, as well as $\mathbb{A}''$ from $\mathbb{A}'$, one has to interchange each labeled operator with its first operand. Here, too, we shall assume that these interchanges are carried out step by step, so that every step consists of a single interchange. Here, as above, we define the **intermediate forms with respect to $\mathscr{T}'$ and $\mathscr{T}''$**. Further, it is easy to notice that every intermediate expression determines uniquely each of the forms $\mathbb{A}$, $\mathbb{A}'$ and $\mathbb{A}''$. Finally, it is obvious that to obtain $\mathbb{A}'$ from $\mathbb{A}$ it suffices first to transform $\mathbb{A}$ into $\mathbb{A}''$ and then $\mathbb{A}''$ into $\mathbb{A}'$. Similarly, $\mathbb{A}$ is obtained from $\mathbb{A}'$.

The axioms and rules of inference in the theories $\mathscr{T}'$ and $\mathscr{T}''$ are obtained from the axioms and rules of inference of the theory $\mathscr{T}$ by means of those explicit changes that are connected with the change of the notions of formula and term. This amounts to the following: the axioms in the theory $\mathscr{T}'$, resp. $\mathscr{T}''$, are those and only those

formulas that correspond, in the above sense, to the axioms of the theory $\mathscr{T}$; and a formula $\mathbb{A}$ of the theory $\mathscr{T}'$, resp. $\mathscr{T}''$, is in the relation of immediate consequence to the formulas $\mathbb{A}_1, \ldots, \mathbb{A}_n$ if and only if the corresponding formulas of the theory $\mathscr{T}$ are in the same relation.

We shall assume that $\mathscr{T}$ is the basic theory, and we shall use the forms of the theories $\mathscr{T}'$ and $\mathscr{T}''$ to denote the corresponding forms of the theory $\mathscr{T}$ (using the forms of the theory $\mathscr{T}'$ to denote the forms of the basic theory makes it easier to understand the content of the context).

Let us note here that any form of each of the theories $\mathscr{T}$, $\mathscr{T}'$ and $\mathscr{T}''$ can be regarded as a denotation of the corresponding form of either of the two other theories. This makes it possible to identify these theories with one another in the following sense. One may, as one wishes, start from any of these three theories and introduce definitions and carry out arguments and proofs in any of them (for example, a proof in one theory can be regarded as a proof in any of these three theories). Moreover, in arguments one can use, instead of the forms of any of these theories, intermediate forms of a fixed type. Further, let us note that the theories $\mathscr{T}$ and $\mathscr{T}''$ can be identified in an even more natural sense. Namely, the forms of the theory $\mathscr{T}$ or (more generally) the forms that are intermediate with respect to the theories $\mathscr{T}$ and $\mathscr{T}''$ can be regarded as simplified versions of forms of the theory $\mathscr{T}''$, obtained by deleting all or several pairs of brackets; conversely, the forms of the theory $\mathscr{T}''$ or (more generally) the mentioned intermediate forms can likewise be regarded as simplified versions of forms of the theory $\mathscr{T}$, obtained by adding several pairs of brackets (in order to bring out clearly the content and the structure of the form).

For definiteness we shall carry out the arguments mainly in the theory $\mathscr{T}$, but it is easy to notice that everything remains valid for $\mathscr{T}'$ as well as for $\mathscr{T}''$, with the obvious changes caused by the change of the notion of form.

We shall call any theory of the kind under consideration a **$\mathscr{T}$-theory** or a **theory of type $\mathscr{T}$**. A formal theory will be called a **generalized $\mathscr{T}$-theory** or a **generalized theory of type $\mathscr{T}$** if its definition is obtained from the definition of a $\mathscr{T}$-theory by weakening the requirement that the set of symbols of each type be finite or countable: it is required instead that the number of symbols of each type be either finite or infinite of any cardinality, in such a way that, instead of each of the sequences (1)–(8) of Section 2.1, a transfinite sequence of any cardinality is taken.

Almost all the definitions and results concerning theories of type $\mathscr{T}$, and the arguments establishing these results, remain valid in the case of generalized $\mathscr{T}$-theories.

2.2.2 Examples

Let us give several examples of theories of type $\mathscr{T}$.

1. Theories $\mathcal{P}_1$, $\mathcal{P}_1'$ and $\mathcal{P}_1''$. The alphabet of the theory $\mathcal{P}_1$ consists of the following symbols:

1) propositional variables $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots$;

2) the basic proper propositional constant f;

3) the basic logical connective $\longrightarrow$ (of weight 2).

By virtue of the general agreement stated above, in the case of $\mathcal{P}_1'$ and $\mathcal{P}_1''$ the alphabet additionally contains the symbols "[" and "]".

Suppose that the propositional variables $\mathbb{P}_0$, $\mathbb{P}_1$ and $\mathbb{P}_2$ are denoted by the meta-symbols p, q and s, respectively. Then the axioms of the theory $\mathcal{P}_1$ are:

I. $\longrightarrow p \longrightarrow q\, p$ (the law of establishing the consequent);

II. $\longrightarrow\longrightarrow s \longrightarrow p\, q \longrightarrow\longrightarrow s\, p \longrightarrow s\, q$ (the law of self-distributivity of implication);

III. $\longrightarrow\longrightarrow\longrightarrow p\, f\, f\, p$ (the law of double negation).

It follows from the general definition of forms, formulas and terms that in the case of the theory $\mathcal{P}_1$ there are no terms, and hence the notions of form and formula coincide.

The rules of inference are the following.

The rule of modus ponens. If $\mathbb{A}$ and $\mathbb{B}$ are any formulas, then $\mathbb{B}$ follows from $\mathbb{A}$ and $\longrightarrow \mathbb{A}\mathbb{B}$.

The rule of substitution. If a is a propositional variable and $\mathbb{A}$ and $\mathbb{B}$ are any formulas, then $(\mathbb{B}/\mathsf{a})\mathbb{A}$ follows from $\mathbb{A}$.[1]

Obviously, in the case of the theory $\mathcal{P}_1'$ the axioms are:

I. $[p \longrightarrow [q \longrightarrow p]]$;

II. $[[s \longrightarrow [p \longrightarrow q]] \longrightarrow [[s \longrightarrow p] \longrightarrow [s \longrightarrow q]]]$;

III. $[[[p \longrightarrow f] \longrightarrow f] \longrightarrow p]$.

2. Theories $\mathcal{F}^1$, $\mathcal{F}^{1\prime}$ and $\mathcal{F}^{1\prime\prime}$. The alphabet of the theory $\mathcal{F}^1$ contains the following symbols:

1) the logical connectives $\longrightarrow$ and $\neg$ (of weights 2 and 1);

2) the logical relational operator sign $\forall$ of weight $(1, 1)$;

3) object variables $\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \ldots$.

In addition to these symbols, the alphabet of the theory $\mathcal{F}^1$ contains all or some of the following symbols (subject to the general agreements stated in the definition of the theory $\mathscr{T}$):

4) propositional variables $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots$;

5) n-place predicate variables $\mathbb{P}_0^n, \mathbb{P}_1^n, \mathbb{P}_2^n, \ldots$ $(n = 1, 2, \ldots)$;

6) basic object constants;

7) basic predicate constants.

[1] The meaning of the expression $(\mathbb{B}/\mathsf{a})\mathbb{A}$ is defined in 2.4.1.

Moreover, the alphabet contains at least one predicate. The class of compound operators consists of the words of the form $\forall \mathsf{a}$, where a is an object variable.

The rules of inference are the rule of modus ponens and the rule of generalization.

The rule of generalization. If $\mathbb{A}$ is any formula and a is an object variable, then $\forall \mathsf{a}\mathbb{A}$ follows from $\mathbb{A}$.

The formulations of the axioms of this theory can be found in [5].

3. Theories $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$. The alphabet of the theory $\mathcal{S}$ contains all the symbols explicitly present in the definition of the theory $\mathscr{T}$; the basic proper two-place predicate constants $=$ and $\in$; the basic logical connectives $\neg$, $\vee$, $\wedge$, $\longrightarrow$, $\longleftrightarrow$ (the weight of the first is 1, and of the others 2); the basic relational logical operator signs $\forall$, $\exists$ and the basic substantive logical operator sign τ (the weight of each of them is $(1, 1)$, and the corresponding quantifiers are the words of the form $\forall \mathsf{a}$, $\exists \mathsf{a}$ and $\tau \mathsf{a}$, where a is any object quantifier letter). In the theory $\mathcal{S}$ the object variables are divided into different categories of variables. Namely, the members with even indices of the sequence of object variables $\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \dots$ (the members are numbered by the natural numbers $0, 1, 2, \dots$) are called **universal object variables**. From the subsequence formed by the remaining members, in the same way, **class variables**, **element variables** and **set variables** are successively singled out. The remaining variables are called **individual**. Likewise, from the sequence $\mathbb{S}'_0, \mathbb{S}'_1, \mathbb{S}'_2, \dots$ of quantifier object constants, the quantifier object constants of different categories are singled out. For every category of object variables and of quantifier object constants an alphabetic enumeration and an alphabetic order are naturally defined.

We need this theory only for introducing illustrative examples when considering general properties of forms. Therefore we do not introduce axioms and rules of inference of this theory.

2.3 General Remarks on Contracting Symbols

2.3.1 Preliminary remarks

In an arbitrary formal mathematical theory, auxiliary symbols called **contracting symbols** are introduced by special metamathematical definitions. They are also called **derived symbols**.

Without the use of contracting symbols we would be forced to write very long forms. The use of forms alone would lead to insurmountable typographical and mental difficulties (we cannot grasp the content of a very long form written out in full, whereas we very often deal with very long forms written in contracted form, so that grasping their content presents no difficulty). For example, in the theoretically very convenient theory of N. Bourbaki [1] the number 1 is introduced as a contracting symbol (as a derived constant) denoting a term that contains several tens of thousands of symbols.

We shall consider contracting symbols of the following types: **derived n-place m-ary operator signs**, $m, n = 1, 2, \dots$, and **derived simple n-place operators**,

$n = 0, 1, 2, \ldots$ Each of them is either logical relational, or logical substantive, or special relational, or special substantive, or logical-special relational, or logical-special substantive. Moreover, every operator sign is either total or partial. Derived n-place operator signs are used for forming derived n-place compound operators (quantifiers) of the corresponding types. Derived simple relational logical operators are also called **derived logical** or **propositional connectives**. Derived nullary simple relational operators are also called **derived propositional constants**. Derived nullary substantive operators are also called **derived object constants**. By definition, the weight of a derived n-place operator is n, and the weight of an n-place m-ary operator sign is the pair (n, m). To every derived n-place logical-special operator sign, resp. operator, σ there is assigned, besides its weight, a **logicality indicator** in the form of some non-empty ordered system of natural numbers $(n_1, \ldots, n_k)$, $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. As the logicality indicator of an operator (quantifier) obtained by applying a derived n-place logical-special operator sign σ one should take the logicality indicator of the operator sign σ. Moreover, to every derived n-place partial operator sign σ there is assigned a **binding indicator** in the form of some non-empty ordered system of natural numbers $(n_1, \ldots, n_k)$, $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. The same system is assigned as the binding indicator to every derived operator (quantifier) obtained by applying the derived operator sign σ (if σ is a logical-special partial operator sign, then both a logicality indicator and a binding indicator are assigned to it).

From the compound terms introduced, terms denoting more general notions are naturally induced. For example, the mention of the term "basic operator" without a direct definition of its meaning is to be understood as an *indirect* definition of this term, regarding it as the general name for all the different types of basic operators introduced. Accordingly, the meaning of the following terms is to be understood: letter, operator sign, operator, constant, special quantifier, basic logical operator, predicate, n-place function, relational substantive operator, etc. Moreover, in compound terms we shall often omit the words "basic" and "derived" if there is no danger of misunderstanding.

We have adopted the convention by which the term "basic operator" denotes only non-nullary operators (unless the contrary follows from the context). We shall not adopt an analogous convention in the case of derived operators (i.e. the term "derived operator" denotes derived operators including the 0-place derived operators).

2.3.2 Free and bound letters

The **scope of an operator** is defined as usual. If σ is a quantifier, then we shall distinguish the content of the terms "scope of the operator σ" and "scope of the quantifier σ" only in the case when σ is a partial quantifier. Namely, in the case when σ is a partial quantifier with binding indicator $(n_1, \ldots, n_k)$, the **scope of the quantifier** σ consists, by definition, of its operands with the numbers $n_1, \ldots, n_k$.

A given occurrence of a letter a in a form $\mathbb{A}$ is, by definition, a **bound occurrence of the letter** a **in the form** $\mathbb{A}$ if this occurrence is an operator letter or it lies in the

scope of a quantifier among whose operator letters a occurs. This notion is generalized as follows. A given occurrence of a letter a in a word $\mathbb{A}$ is, by definition, a **bound occurrence of the letter** a **in the word** $\mathbb{A}$ if there is a part $\mathbb{A}_1$ of the word $\mathbb{A}$ which is a form and the occurrence under consideration is a bound occurrence of the letter a in the form (in the part) $\mathbb{A}_1$. Otherwise the occurrence under consideration is called a **free occurrence of the letter** a **in the word** $\mathbb{A}$. It is obvious that a given occurrence of a letter a in a form $\mathbb{A}$ is a free occurrence of a in $\mathbb{A}$ if and only if it is not a bound occurrence of a in $\mathbb{A}$. It is likewise obvious that the only occurrence of the letter a in the word a is a free occurrence of a in the word a. Let $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m\mathbb{A}_1 \cdots \mathbb{A}_n$ be such a part of the word $\mathbb{A}$ (in particular, of the form $\mathbb{A}$) that it (regarded as a word) is a form, where σ is an operator sign, $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m$ is an n-place quantifier with binding indicator $(n_1, \ldots, n_k)$, the $\mathsf{a}_1, \ldots, \mathsf{a}_m$ are operator letters, and $\mathbb{A}_1, \ldots, \mathbb{A}_n$ are forms. Then the occurrences of the letters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ in the operator $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m$, and the free occurrences of these letters in the parts $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_k}$ of the word (in particular, of the form) $\mathbb{A}$, and only they, are said to be **bound by the operator sign** σ (we mean the occurrence of the operator sign σ). Moreover, by definition, the occurrence of the letter a_k in the operator $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m$ as its $(k+1)$-th member, and the free occurrences of the letter a_k in the above-mentioned parts $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_k}$, are regarded as k**-th letters bound by the operator sign** σ (we mean the occurrences of the letter a_k); obviously, one and the same occurrence in a form $\mathbb{A}$ of some letter may, for $k \neq j$, be both a k-th letter bound by an operator sign σ and a j-th letter bound by the operator sign σ. Finally, a letter a is called a **free letter** of a form (word) $\mathbb{A}$, resp. a **bound letter** of a form (word) $\mathbb{A}$, if the letter a has a free, resp. a bound, occurrence in the form (word) $\mathbb{A}$.

2.3.3 Congruence and variance of forms

In connection with the notions of free and bound occurrences of letters in forms, the congruence and the variance of forms are introduced; they are defined as follows.

By a **group of bound occurrences of letters of a form** $\mathbb{A}$ we mean a maximal set $\{\mathsf{a}_1, \ldots, \mathsf{a}_k\}$ of occurrences of one and the same letter in the form $\mathbb{A}$ having the following property: (a) there is an occurrence σ of some operator sign such that each element of the set $\{\mathsf{a}_1, \ldots, \mathsf{a}_k\}$ is an occurrence bound by the operator sign σ (we mean the occurrence of the operator sign). Here the maximality of the set $\{\mathsf{a}_1, \ldots, \mathsf{a}_k\}$ means that it is not a proper part of a set having the property (a). By a **group of free (occurrences of) letters of a form** $\mathbb{A}$ we mean the non-empty set of all free occurrences of one and the same quantifier letter in the form $\mathbb{A}$. As the general name for a group of bound letters of a form and a group of free letters of a form we shall use the term **group of (occurrences of) letters of a form**. Thus we have groups of letters of a form of two *kinds*. The **type** of a group of letters of a form is determined by its kind and by the type of the corresponding letter (i.e. by definition, two groups of letters of a form have one and the same type if and only if the corresponding letters have one and the same type and, moreover, either each of them is a group of free letters or each of them is a group of bound letters).

A one-to-one correspondence f between the sets of constituent symbols of words $\mathbb{A}$ and $\mathbb{B}$ of one and the same length l is called **canonical** if to the i-th symbol of the word $\mathbb{A}$ there corresponds the i-th symbol of the word $\mathbb{B}$, $i = 1, 2, \ldots, l$.

We shall say that a form $\mathbb{A}$ is a **variant** of a form $\mathbb{B}$, or equivalently that $\mathbb{A}$ and $\mathbb{B}$ are **mutually variant forms**, if they have one and the same length l (i.e. contain the same number of (occurrences of) symbols) and under the canonical correspondence between the sets of their symbols we have: (a) to any symbol of the form $\mathbb{A}$ that is not a quantifier letter there corresponds the same symbol (of $\mathbb{B}$); (b) to every group of free letters of the form $\mathbb{A}$ there corresponds a group of free letters of the form $\mathbb{B}$ of the same type; (c) to every group of bound letters of the form $\mathbb{A}$ there corresponds a group of bound letters of the form $\mathbb{B}$ of the same type. We say that a form $\mathbb{A}$ is **congruent** to a form $\mathbb{B}$, or equivalently that the forms $\mathbb{A}$ and $\mathbb{B}$ are **mutually congruent** (and write $\mathbb{A} \simeq \mathbb{B}$), if they have one and the same length l and under the canonical correspondence between the sets of their symbols condition (c) holds together with the following condition: (d) to any symbol of the form $\mathbb{A}$ that is not a bound occurrence of a letter there corresponds the same symbol.

In the case when constants are not used as operator letters, in the given definition of mutual congruence of forms the word "letter" may be replaced by the word "variable". Obviously, (d) $\Rightarrow$ (a) $\wedge$ (b) and hence mutually congruent forms are mutually variant forms.

A **simultaneous replacement of groups of letters**, resp. **of groups of bound letters**, resp. **of groups of free letters**,

$$\{\mathsf{a}_1^1, \ldots, \mathsf{a}_{k_1}^1\}, \ \ldots, \ \{\mathsf{a}_1^n, \ldots, \mathsf{a}_{k_n}^n\},$$

is, by definition, a replacement of every occurrence in these groups by a letter of the same type, in such a way that the elements of one and the same group are replaced by one and the same letter, and the word obtained as the result of the replacement is a form, and in this form every set of occurrences of letters replacing occurrences of letters of one and the same group is a group of letters (of the same type). In the case $n = 1$, instead of the terms just introduced we shall also use the following terms: **replacement of a group of letters**, **replacement of a group of bound letters**, **replacement of a group of free letters**. A simultaneous replacement of groups of letters, resp. of groups of bound letters, resp. of groups of free letters, can be carried out by successive application of replacements of a group of letters, resp. of a group of bound letters, resp. of a group of free letters (this is proved by the method by which simultaneous substitutions are reduced to sequences of simple substitutions). It is easy to see that two forms are mutually variant, resp. mutually congruent, if and only if each of them (one of them) is obtained from the other by a simultaneous replacement of several groups of letters, resp. groups of bound letters. It is also easy to see that a group of bound letters of a form $\mathbb{A}$ is a group of bound letters of a part $\mathbb{A}_1$ of the form $\mathbb{A}$ if and only if at least one element of this group is a bound letter of the part $\mathbb{A}_1$. It is also obvious that a group of letters is uniquely determined by any of its elements. Each element of a group of letters is called a **representative** of this

group of letters. A representative of a group of bound letters is called the **canonical representative** if it is the operator letter closest to the corresponding operator sign. Obviously, every group of bound letters has a unique canonical representative.

It is easy to see that the corresponding parts $\mathbb{A}_1$ and $\mathbb{B}_1$ of mutually congruent forms $\mathbb{A}$ and $\mathbb{B}$ are mutually variant forms. Moreover, any group of free letters of the part $\mathbb{A}_1$ is either a proper part of a group of bound letters of the form $\mathbb{A}$ located in the part $\mathbb{A}_1$, or a part of a group of free letters of the form $\mathbb{A}$ located in the part $\mathbb{A}_1$. Finally, it is obvious that the relation of congruence, resp. of variance, has the properties of reflexivity, symmetry and transitivity.

2.4 Replacements and Substitutions

2.4.1 Substitutions

Below we shall need the substitution signs $(— /_: —)$, $(— /_. —)$ and $(—/—)$, which are used as follows.

If $\mathbb{A}$ is a form, $\mathbb{B}_1, \ldots, \mathbb{B}_n$ are words and $\mathsf{b}_1, \ldots, \mathsf{b}_n$ are distinct letters, then

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n /_: \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}, \tag{1}$$

resp.

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n /_. \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A} \tag{2}$$

(read: $\mathbb{B}_1, \ldots, \mathbb{B}_n$ *replace all occurrences of the letters* $\mathsf{b}_1, \ldots, \mathsf{b}_n$ *in* $\mathbb{A}$, resp. $\mathbb{B}_1, \ldots, \mathbb{B}_n$ *replace the free occurrences of the letters* $\mathsf{b}_1, \ldots, \mathsf{b}_n$ *in* $\mathbb{A}$) denotes the word that is obtained if, for each $i \in \{1, 2, \ldots, n\}$, we substitute the word $\mathbb{B}_i$ for each occurrence, resp. each free occurrence, of the letter b_i in $\mathbb{A}$. One says that (1), resp. (2), is the **result of the simultaneous substitution of the words** $\mathbb{B}_1, \ldots, \mathbb{B}_n$ **in** $\mathbb{A}$ **for all occurrences**, resp. **for the free occurrences**, **of the letters** $\mathsf{b}_1, \ldots, \mathsf{b}_n$ **in** $\mathbb{A}$. One also says that $(\mathbb{B}_i /_: \mathsf{b}_i)\mathbb{A}$, resp. $(\mathbb{B}_i /_. \mathsf{b}_i)\mathbb{A}$, is the **result of the substitution of the word** $\mathbb{B}_i$ **in** $\mathbb{A}$ **for all occurrences**, resp. **for the free occurrences**, **of the letter** b_i **in** $\mathbb{A}$. A substitution of the last kind is also called a **simple substitution** for all occurrences, resp. for free occurrences. The expression $(\mathbb{B}_i /_: \mathsf{b}_i)\mathbb{A}$ is read: $\mathbb{B}_i$ replaces all occurrences of the letter b_i in $\mathbb{A}$, and $(\mathbb{B}_i /_. \mathsf{b}_i)\mathbb{A}$ is read: $\mathbb{B}_i$ replaces the free occurrences of the letter b_i in $\mathbb{A}$.

Thus we have become acquainted with the notion of a simultaneous substitution in a form $\mathbb{A}$ for all occurrences, resp. for the free occurrences, of the letters $\mathsf{b}_1, \ldots, \mathsf{b}_n$ in the case when $\mathsf{b}_1, \ldots, \mathsf{b}_n$ are pairwise distinct. If the sequence $\mathsf{b}_1, \ldots, \mathsf{b}_n$ has repeated members, then it is convenient to generalize these notions of simultaneous substitution as follows. By definition,

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n /_: \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A} = (\mathbb{B}_{i_1}, \ldots, \mathbb{B}_{i_k} /_: \mathsf{b}_{i_1}, \ldots, \mathsf{b}_{i_k})\mathbb{A};$$
$$(\mathbb{B}_1, \ldots, \mathbb{B}_n /_. \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A} = (\mathbb{B}_{i_1}, \ldots, \mathbb{B}_{i_k} /_. \mathsf{b}_{i_1}, \ldots, \mathsf{b}_{i_k})\mathbb{A};$$

where $\mathsf{b}_{i_1}, \ldots, \mathsf{b}_{i_k}$ is the sequence obtained from the sequence $\mathsf{b}_1, \ldots, \mathsf{b}_n$ by deleting the repeated members (a member of a sequence is called *repeated* if it is equal to some preceding member; $\eqcirc$ is the sign of graphic equality).

Let $\mathbb{A}$ be a form, let $\mathsf{b}_1, \ldots, \mathsf{b}_n$ be distinct letters, and for each $i \in \{1, 2, \ldots, n\}$ let b_i and $\mathbb{B}_i$ be either letters of one and the same type or forms of one and the same type. Then

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}$$

(read: $\mathbb{B}_1, \ldots, \mathbb{B}_n$ *replace the letters* $\mathsf{b}_1, \ldots, \mathsf{b}_n$ *in* $\mathbb{A}$) denotes, by definition, the last member of the sequence of forms

$$\mathbb{A}_0 \eqcirc \mathbb{A},\ \mathbb{A}_1,\ \ldots,\ \mathbb{A}_k \eqcirc (\mathbb{B}_1, \ldots, \mathbb{B}_n\ /_{\cdot}\ \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}_{k-1},$$

uniquely determined by the following condition. For each $i \in \{1, 2, \ldots, k-1\}$ the form $\mathbb{A}_i$ is obtained from $\mathbb{A}_{i-1}$ as follows. Find in the form $\mathbb{A}_{i-1}$ the first such occurrence b of a letter among the free occurrences of the letters $\mathsf{b}_1, \ldots, \mathsf{b}_n$ that lies in the scope of a quantifier among whose operator letters there is a free letter of the word $\mathbb{B}_j$, where j is the natural number such that b is b_j. Among such quantifiers choose the quantifier located to the left of all the others, and among the operator letters of the last quantifier that have the indicated property choose the letter located to the left of all the others. Let it be an occurrence a of some letter (such an occurrence a exists in every member of the sequence $\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_{k-2}$ and does not exist in $\mathbb{A}_{k-1}$). After this, to obtain $\mathbb{A}_i$ it suffices to replace every member of the group of bound letters containing the occurrence a by the letter c, where c is the first letter in the alphabetic enumeration of the letters of the type of the letter a that is not a free letter of any of the words $\mathbb{B}_1, \ldots, \mathbb{B}_n$; the replacement carried out is thus a replacement of the mentioned group of bound letters.

If we take into account the definition of the term "the word $\mathbb{B}$ is free for the letter a in the form $\mathbb{A}$", given below in this same section, it is easy to see that each of the forms $\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_{k-1}$ is congruent to the form $\mathbb{A}$, and moreover $\mathbb{A}_{k-1}$ is a form congruent to $\mathbb{A}$ for which the following holds.

Condition (a_n). The word $\mathbb{B}_i$ is free for b_i in $\mathbb{A}_{k-1}$, $i = 1, 2, \ldots, n$.

It is easy to notice that if the condition (a_n) holds for $\mathbb{A}$, then $k = 1$ and, consequently,

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A} \eqcirc (\mathbb{B}_1, \ldots, \mathbb{B}_n\ /_{\cdot}\ \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}.$$

In the general case, when the letters $\mathsf{b}_1, \ldots, \mathsf{b}_n$ are not required to be distinct, we have, by definition,

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A} \eqcirc (\mathbb{B}_{i_1}, \ldots, \mathbb{B}_{i_k}/\mathsf{b}_{i_1}, \ldots, \mathsf{b}_{i_k})\mathbb{A},$$

where $\mathsf{b}_{i_1}, \ldots, \mathsf{b}_{i_k}$ is the sequence obtained from the sequence $\mathsf{b}_1, \ldots, \mathsf{b}_n$ by deleting the repeated members.

We call the form $(\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}$ the **result of the simultaneous substitution of the words** $\mathbb{B}_1, \ldots, \mathbb{B}_n$ **in the form** $\mathbb{A}$ **for the letters** $\mathsf{b}_1, \ldots, \mathsf{b}_n$. The form

$(\mathbb{B}_i/\mathsf{b}_i)\mathbb{A}$ is also called the **result of the simple substitution of the word $\mathbb{B}_i$ in the form $\mathbb{A}$ for the letter b_i**.

Now the following is easily proved.

Theorem 2.1. 1. *Let $\mathbb{A}$ and $\mathbb{B}$ be mutually congruent forms and let $\mathsf{a}_1, \ldots, \mathsf{a}_n$ be letters. Suppose that for each $i \in \{1, 2, \ldots, n\}$ either a_i, $\mathbb{A}_i$ and $\mathbb{B}_i$ are letters of one and the same type and $\mathbb{A}_i$ and $\mathbb{B}_i$ are one and the same letter, or a_i, $\mathbb{A}_i$ and $\mathbb{B}_i$ are forms of one and the same type and $\mathbb{A}_i \simeq \mathbb{B}_i$. Then*

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A} \simeq (\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{B}.$$

In particular, if for each $i \in \{1, 2, \ldots, n\}$ the word $\mathbb{A}_i$ is free for a_i in $\mathbb{A}$ and the word $\mathbb{B}_i$ is free for a_i in $\mathbb{B}$, then

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n \,/_{\cdot}\, \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A} \simeq (\mathbb{B}_1, \ldots, \mathbb{B}_n \,/_{\cdot}\, \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{B}.$$

2. *Let $\mathbb{A}$ be a form, let $\mathsf{a}_1, \ldots, \mathsf{a}_n$ be letters, and for each $i \in \{1, 2, \ldots, n\}$ let $\mathbb{A}_i$ and a_i be letters of one and the same type or forms of one and the same type. Further, let $\mathsf{b}_1, \ldots, \mathsf{b}_m$ be the enumeration of all those bound letters of the form $\mathbb{A}$ each of which is a free letter of at least one of the words $\mathbb{A}_1, \ldots, \mathbb{A}_n$. Finally, let $\mathsf{d}_1, \ldots, \mathsf{d}_m$ be the sequence of letters uniquely determined by the following condition: d_i is the first letter in the alphabetic enumeration of the letters whose type coincides with the type of the letter b_i and which is not found in any member of the sequence*

$$\mathbb{A}, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{d}_1, \ldots, \mathsf{d}_{i-1} \qquad (i = 1, 2, \ldots, m).$$

Denote by $\mathbb{A}^*$ the form obtained as the result of substituting the letters $\mathsf{d}_1, \ldots, \mathsf{d}_m$ in $\mathbb{A}$ for the bound occurrences of the letters $\mathsf{b}_1, \ldots, \mathsf{b}_m$, respectively. Then $\mathbb{A} \simeq \mathbb{A}^*$, the word $\mathbb{A}_i$ is free for a_i in $\mathbb{A}^*$, and consequently

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A} \simeq (\mathbb{A}_1, \ldots, \mathbb{A}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A}^* = (\mathbb{A}_1, \ldots, \mathbb{A}_n \,/_{\cdot}\, \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A}^*.$$

Thus we have become acquainted with the expressions $(— /_: —)$, $(— /_{\cdot} —)$ and $(—/—)$ and with the rules of their use for writing the results of substitutions. The substitutions corresponding to these expressions we shall call, respectively, **substitution of type** $(— /_: —)$, **substitution of type** $(— /_{\cdot} —)$ and **substitution of type** $(—/—)$. Instead of these terms we shall also use the terms $(— /_: —)$-*substitution*, $(— /_{\cdot} —)$-*substitution* and $(—/—)$-*substitution*.

2.4.2 The property of relative freedom

Let $\mathbb{A}$ be a form, $\mathbb{A}_1$ a part of the word $\mathbb{A}$, a a letter and $\mathbb{B}$ a word, where $\mathbb{B}$ and $\mathbb{A}_1$ are letters or forms of one and the same type. We shall say that the word $\mathbb{B}$ is **free**, resp. **completely free**, **for $\mathbb{A}_1$ in $\mathbb{A}$** if $\mathbb{A}_1$ does not lie in the scope of a quantifier among whose operator letters there occurs a free letter of the word $\mathbb{B}$ that has no free

occurrence in $\mathbb{A}_1$, resp. among whose operator letters there occurs a free letter of the word $\mathbb{B}$. In other words, "$\mathbb{B}$ is completely free (resp. free) for $\mathbb{A}_1$ in $\mathbb{A}$" means that in the form obtained by substituting in $\mathbb{A}$ the word $\mathbb{B}$ in place of the part $\mathbb{A}_1$ all the free letters of the substituted occurrence $\mathbb{B}$ will remain free (resp. all the free letters of the substituted occurrence $\mathbb{B}$ that are not free letters of the part (word) $\mathbb{A}_1$ will remain free). Moreover, we shall say that the word $\mathbb{B}$ is **free for the letter** a **in the form** $\mathbb{A}$ if the word $\mathbb{B}$ is free in the form $\mathbb{A}$ for every free occurrence of the letter a in $\mathbb{A}$. The content of the last definition does not change if, instead of freedom, we require complete freedom of $\mathbb{B}$. Obviously, the word $\mathbb{B}$ is free for the letter a in the form $\mathbb{A}$ if and only if every free letter of the word $\mathbb{B}$ is free in the form $\mathbb{A}$ for every free occurrence of the letter a in $\mathbb{A}$.

The following follows directly from the definitions.

Let $\mathbb{A}$ be a form, let $\mathsf{b}_1, \ldots, \mathsf{b}_n$ be distinct letters, and for each $i \in \{1, 2, \ldots, n\}$ let $\mathbb{B}_i$ and b_i be letters or forms of one and the same type. Then $\mathbb{A}$ and $(\mathbb{B}_1, \ldots, \mathbb{B}_n \,/_. \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}$ are forms of one and the same type and, moreover, in this case the following two conditions are equivalent.

(a_n) For each $i \in \{1, 2, \ldots, n\}$ the word $\mathbb{B}_i$ is free for the letter b_i in the form $\mathbb{A}$.

(a'_n) If $i \in \{1, 2, \ldots, n\}$ and $\mathbb{B}'_i$ is an occurrence in the form $(\mathbb{B}_1, \ldots, \mathbb{B}_n \,/_. \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}$ obtained as the result of substituting the word $\mathbb{B}_i$ for some free occurrence of the letter b_i in the form $\mathbb{A}$, and C_i is some occurrence of a letter C in $\mathbb{B}_i$, then C_i is a bound occurrence of the letter C in $\mathbb{B}_i$ if and only if the occurrence of the letter C in $\mathbb{B}'_i$ corresponding to C_i is a bound occurrence of the letter C in the form $(\mathbb{B}_1, \ldots, \mathbb{B}_n \,/_. \mathsf{b}_1, \ldots, \mathsf{b}_n)\mathbb{A}$.

2.4.3 The scope of an operator letter and related notions

Let $\mathbb{A}$ be a form, let $\mathbb{A}_1$ be a part of the form $\mathbb{A}$, resp. an occurrence of some letter b in the form $\mathbb{A}$, and let a be an operator letter of a quantifier σ of the form $\mathbb{A}$ (we mean an occurrence of the quantifier). We shall say that the part $\mathbb{A}_1$, resp. the occurrence $\mathbb{A}_1$ of the letter b under consideration, **lies in the scope of the operator letter** a if $\mathbb{A}_1$ is an operator letter of the quantifier σ, or $\mathbb{A}_1$ lies in the scope of the quantifier σ and, moreover, $\mathbb{A}_1$ is not an operator letter and does not lie in the scope of such a quantifier σ' which lies in the scope of the quantifier σ and among whose operator letters there occurs the letter a. If, in addition, a (regarded as a letter) is a free letter of the part $\mathbb{A}_1$, resp. is $\mathbb{A}_1$, then we shall also say that the operator letter a **acts on** $\mathbb{A}_1$. Otherwise we shall say that the occurrence a of the letter does not act on $\mathbb{A}_1$ (it is understood that if an occurrence of a letter is not an operator letter, then this occurrence of the letter acts on no part, resp. on no occurrence of a letter). Moreover, we shall say that the quantifier σ and the operator sign of the quantifier σ (we mean an occurrence of the quantifier and an occurrence of its operator sign) **act on** $\mathbb{A}_1$ if some operator letter of the quantifier σ acts on $\mathbb{A}_1$. Obviously, an operator letter a acts on every element of the group of bound letters containing a and acts on no other occurrence of a letter.

Let $\mathbb{A}$ and $\mathbb{B}$ be forms, let $\mathbb{A}_1$ be a part of the form $\mathbb{A}$ whose type coincides with the type of the form $\mathbb{B}$, and let a be an operator letter of a quantifier σ of the form $\mathbb{A}$ (we mean an occurrence of the quantifier). We shall say that the operator letter a **acts on the form** $\mathbb{B}$ **located in place of the part** $\mathbb{A}_1$ if the form $\mathbb{B}$ contains a free occurrence of the letter a and the part $\mathbb{A}_1$ of the form $\mathbb{A}$ lies in the scope of the operator letter a.

2.4.4 Replacements

Now we become acquainted with the signs $(— //_: —)$ and $(— // —)$. The first is called the **sign of direct replacement**, and the second the **sign of replacement**.

The sign $(— //_: —)$ is used as follows. If $\mathbb{B}_1, \ldots, \mathbb{B}_n, \mathbb{A}$ are words and $\mathbb{A}_1, \ldots, \mathbb{A}_n$ are (pairwise) non-overlapping parts of the word $\mathbb{A}$, then

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n //_: \mathbb{A}_1, \ldots, \mathbb{A}_n)\mathbb{A} \tag{1}$$

(read: $\mathbb{B}_1, \ldots, \mathbb{B}_n$ *directly replace the parts* $\mathbb{A}_1, \ldots, \mathbb{A}_n$ *in* $\mathbb{A}$) denotes the word obtained from $\mathbb{A}$ by replacing the parts $\mathbb{A}_1, \ldots, \mathbb{A}_n$ by the words $\mathbb{B}_1, \ldots, \mathbb{B}_n$, respectively. We shall say that (1) is the **result of the direct replacement of the parts** $\mathbb{A}_1, \ldots, \mathbb{A}_n$ **of the word** $\mathbb{A}$ **by the words** $\mathbb{B}_1, \ldots, \mathbb{B}_n$, respectively. Sometimes (1) is called the **result of the direct (simultaneous) substitution in the word** $\mathbb{A}$ **of the words** $\mathbb{B}_1, \ldots, \mathbb{B}_n$ **in place of the parts** $\mathbb{A}_1, \ldots, \mathbb{A}_n$.

The sign $(— // —)$ is used as follows. Let $\mathbb{B}_1, \ldots, \mathbb{B}_n, \mathbb{A}$ be forms, and let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be non-overlapping parts of the form $\mathbb{A}$, where for each $i \in \{1, 2, \ldots, n\}$ the forms $\mathbb{A}_i$ and $\mathbb{B}_i$ are forms of one and the same type. Then

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n // \mathbb{A}_1, \ldots, \mathbb{A}_n)\mathbb{A}$$

(read: $\mathbb{B}_1, \ldots, \mathbb{B}_n$ *replace the parts* $\mathbb{A}_1, \ldots, \mathbb{A}_n$ *in* $\mathbb{A}$) denotes the last member of the sequence

$$\mathbb{A}'_0 = \mathbb{A},\ \mathbb{A}'_1,\ \ldots,\ \mathbb{A}'_k = (\mathbb{B}_1, \ldots, \mathbb{B}_n //_: \mathbb{A}^1_{k-1}, \ldots, \mathbb{A}^n_{k-1})\mathbb{A}'_{k-1}.$$

Here the sequence $\mathbb{A}'_1, \ldots, \mathbb{A}'_{k-1}$ and the parts $\mathbb{A}^j_i$ of the forms $\mathbb{A}'_i$ ($j = 1, 2, \ldots, n$; $i = 0, 1, 2, \ldots, k-1$) are uniquely determined by the algorithmic process defined by the following rule.

Let $\mathbb{A}^j_0 = \mathbb{A}_j$ ($j = 1, 2, \ldots, n$). Suppose that the following condition (a_m), depending on the positive natural number m, is satisfied.

Condition (a_m). The form $\mathbb{A}'_{m-1}$ and its non-overlapping parts $\mathbb{A}^1_{m-1}, \ldots, \mathbb{A}^n_{m-1}$ are defined, and, moreover, among these parts there is a part $\mathbb{A}^\alpha_{m-1}$ that lies in the scope of a quantifier among whose operator letters there occurs a free letter of the form $\mathbb{B}_\alpha$ that is not a free letter of the form $\mathbb{A}^\alpha_{m-1}$ (the first part of the condition (a_1) is satisfied).

Suppose that $\mathbb{A}^\alpha_{m-1}$ is located to the left of the other parts among $\mathbb{A}^1_{m-1}, \ldots, \mathbb{A}^n_{m-1}$ having the indicated property. Further, let σ be a quantifier having the indicated

property with respect to $\mathbb{A}^{\alpha}_{m-1}$ and located to the left of the other such quantifiers, and let a be an operator letter of the quantifier σ having the indicated property and located to the left of all the other such operator letters of the quantifier σ. Replace by the letter c every member of the group of bound letters containing the occurrence of the letter a under consideration, where c is the first letter in the alphabetic enumeration of the letters whose type coincides with the type of the letter a that is not a free letter of any of the forms $\mathbb{B}_1, \ldots, \mathbb{B}_n$; this replacement is carried out as a replacement of the mentioned group of bound letters. Denote by $\mathbb{A}'_m, \mathbb{A}^1_m, \ldots, \mathbb{A}^n_m$ the form and its non-overlapping parts into which the form $\mathbb{A}'_{m-1}$ and its parts $\mathbb{A}^1_{m-1}, \ldots, \mathbb{A}^n_{m-1}$ are transformed as the result of the replacement of the mentioned group of bound letters. To the metavariable m we give successively the values $1, 2, 3, \ldots$ and, if the corresponding condition (a_m) is satisfied, we find the corresponding form and its parts. Otherwise the algorithmic process stops.

It is easy to see that the algorithmic process defined by the above rule terminates (k is the first natural number for which the condition (a_k) is not satisfied).

Obviously, $\mathbb{A}'_0, \mathbb{A}'_1, \ldots, \mathbb{A}'_{k-1}$ are forms congruent to the form $\mathbb{A}$ and, moreover, if $l \in \{0, 1, \ldots, k-1\}$, then the following condition holds only in the case $l = k - 1$.

Condition (b_l). The form $\mathbb{B}_i$ is free for $\mathbb{A}^i_l$ in $\mathbb{A}'_l$, $i = 1, 2, \ldots, n$.

It is easy to see that if the condition (b_0) is satisfied, then $k = 1$ and, consequently,

$$(\mathbb{B}_1, \ldots, \mathbb{B}_n \mathbin{//} \mathbb{A}_1, \ldots, \mathbb{A}_n)\mathbb{A} = (\mathbb{B}_1, \ldots, \mathbb{B}_n \mathbin{//_:} \mathbb{A}_1, \ldots, \mathbb{A}_n)\mathbb{A}.$$

The form $(\mathbb{B}_1, \ldots, \mathbb{B}_n \mathbin{//} \mathbb{A}_1, \ldots, \mathbb{A}_n)\mathbb{A}$ is called the **result of the simultaneous replacement of the forms** $\mathbb{B}_1, \ldots, \mathbb{B}_n$ **in** $\mathbb{A}$ **in place of the parts** $\mathbb{A}_1, \ldots, \mathbb{A}_n$. The form $(\mathbb{B}_i \mathbin{//} \mathbb{A}_i)\mathbb{A}$ is also called the **result of the simple replacement of the form** $\mathbb{B}_i$ **in** $\mathbb{A}$ **in place of the part** $\mathbb{A}_i$. Instead of these terms we shall also use the terms obtained from them by replacing the word "replacement" by the word "substitution".

Remark. It is easy to see that in the case $n = 1$ we have $\mathbb{A}^1_{k-1} = \mathbb{A}_1$.

Suppose that two or more theories of type $\mathscr{T}$ are considered simultaneously, each of which is denoted by the symbol $\mathscr{T}$ or by the symbol $\mathscr{T}$ with indices, and that it is possible for the forms of one of the theories under consideration to be regarded as denotations of forms of another theory. Then, in order to avoid misunderstandings, when introducing replacements and substitutions in one of the theories under consideration, instead of the expressions $(—/—)$, $(— \mathbin{//} —)$, $(— \mathbin{/_:} —)$, etc. their full analogues will be used, obtained by supplying the symbols "/" and "//" in these expressions with the same indices as the symbol denoting the theory. In more complicated cases, to avoid misunderstandings, the symbols "/" and "//" will be supplied, as lower indices, with the symbols denoting the theory. Thus, for example, to introduce replacements and substitutions in the theory of type $\mathscr{T}$ denoted by the symbol $\widetilde{\mathscr{T}}_a$, expressions of the following kinds will be used: $(—/_{\widetilde{\mathscr{T}}_a}—)$, $(—//_{\widetilde{\mathscr{T}}_a}—)$, $(—/_{\widetilde{\mathscr{T}}_a:}—)$, etc. Likewise, to introduce replacements and substitutions in the theory of type $\mathscr{T}$ denoted by the symbol $\mathcal{P}_1$, expressions of the following kinds will be used: $(—/_{\mathcal{P}_1}—)$, $(—//_{\mathcal{P}_1}—)$, $(—/_{\mathcal{P}_1:}—)$, etc.

2.4.5 Properties of substitutions and replacements

Theorem 2.2. *Let $\mathbb{A}, \mathbb{B}, \mathbb{A}_2, \mathbb{B}_2$ be forms such that $\mathbb{A} \simeq \mathbb{B}$ and $\mathbb{A}_2 \simeq \mathbb{B}_2$, and let $\mathbb{A}_1$ and $\mathbb{B}_1$ be corresponding parts of the forms $\mathbb{A}$ and $\mathbb{B}$ whose type coincides with the type of the forms $\mathbb{A}_2$ and $\mathbb{B}_2$. Further, let $\mathsf{a}_1, \ldots, \mathsf{a}_n$ be such an enumeration of the canonical representatives of (certain) groups of bound letters of the form $\mathbb{A}$ that the canonical representative a of a group of bound letters of the form $\mathbb{A}$ is a member of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ if and only if a acts on the form $\mathbb{A}_2$ located in place of $\mathbb{A}_1$, and, moreover, the order among these representatives in the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ is the same as their order in the form $\mathbb{A}$.*

Furthermore, let $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$ be the occurrences of letters in $\mathbb{B}$ corresponding to the occurrences $\mathsf{a}_1, \ldots, \mathsf{a}_n$. We introduce the following notation: α'_i, resp. α_i, is the occurrence a'_i, resp. a_i, regarded as a letter, $i = 1, 2, \ldots, n$ (the letters $\alpha_1, \ldots, \alpha_n$ are pairwise distinct); $\mathbb{B}'_2$ is some form congruent to the form $\mathbb{B}_2$ and satisfying the condition: for each $i \in \{1, 2, \ldots, n\}$ the letter α'_i is free for α_i in $\mathbb{B}'_2$;

$$\mathbb{B}^*_2 \rightleftharpoons (\alpha'_1, \ldots, \alpha'_n \,/_{\cdot}\, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2; \qquad \mathbb{A}' \rightleftharpoons (\mathbb{A}_2 \,//_{:}\, \mathbb{A}_1)\mathbb{A}; \qquad \mathbb{B}' \rightleftharpoons (\mathbb{B}^*_2 \,//_{:}\, \mathbb{B}_1)\mathbb{B}$$

(as $\mathbb{B}^*_2$ one may take $(\alpha'_1, \ldots, \alpha'_n/\alpha_1, \ldots, \alpha_n)\mathbb{B}_2$).

Under these conditions we have:

1. *Suppose that* (a) *the sequence of canonical representatives associated with the forms $\mathbb{B}$ and $\mathbb{B}^*_2$ and the part $\mathbb{B}_1$ of the form $\mathbb{B}$, and defined analogously to the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$, coincides with the sequence $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$, and* (b) *every free occurrence of any of the letters $\alpha'_1, \ldots, \alpha'_n$ in the form*

$$\mathbb{B}^*_2 \rightleftharpoons (\alpha'_1, \ldots, \alpha'_n \,/_{\cdot}\, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2$$

is a substituted occurrence. Then $\mathbb{A}' \simeq \mathbb{B}'$.

2. *Suppose that condition* (a) *is satisfied and the sequence $\alpha_1, \ldots, \alpha_n$ coincides with the sequence $\alpha'_1, \ldots, \alpha'_n$. Then $\mathbb{B}^*_2 \rightleftharpoons \mathbb{B}'_2$ and $\mathbb{A}' \simeq \mathbb{B}'$. Moreover, as the form $\mathbb{B}'_2$, and hence as the form $\mathbb{B}^*_2$, one may take $\mathbb{B}_2$, i.e. $(\mathbb{A}_2 \,//_{:}\, \mathbb{A}_1)\mathbb{A} \simeq (\mathbb{B}_2 \,//_{:}\, \mathbb{B}_1)\mathbb{B}$.*

3. *Suppose that the form $\mathbb{A}_2$ is free for $\mathbb{A}_1$ in the form $\mathbb{A}$, the form $\mathbb{B}^*_2$ is free for $\mathbb{B}_1$ in the form $\mathbb{B}$, every free letter of the part $\mathbb{A}_1$ that is a bound letter in $\mathbb{A}$ is a free letter of the form $\mathbb{A}_2$, and every free letter of the part $\mathbb{B}_1$ that is a bound letter in $\mathbb{B}$ is a free letter of the form $\mathbb{B}^*_2$. Then condition* (a) *will be satisfied. Consequently, if condition* (b) *is satisfied in addition, then $\mathbb{A}' \simeq \mathbb{B}'$.*

Remark. As is easily seen, the satisfaction of conditions (a) and (b) does not depend on the choice of the form $\mathbb{B}'_2$. In the case when condition (a) is satisfied (or, more generally, when $\alpha'_1, \ldots, \alpha'_n$ are pairwise distinct letters), condition (b) is equivalent to the following condition.

Condition (b*). $\mathbb{B}^*_2$ and $\mathbb{B}'_2$ are mutually variant forms. In other words: by the substitution corresponding to the expression $(\alpha'_1, \ldots, \alpha'_n \,/_{\cdot}\, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2$ a simultaneous replacement of the groups of free letters of the form $\mathbb{B}'_2$ corresponding to the letters $\alpha_1, \ldots, \alpha_n$ is carried out.

Let us note here that the mutual variance of the forms $\mathbb{B}^*_2$ and $\mathbb{B}'_2$ is a necessary

and sufficient condition for the mutual variance of the forms $\mathbb{B}_2^*$ and $\mathbb{A}_2$; and the mutual variance of the forms $\mathbb{B}_2^*$ and $\mathbb{A}_2$ is, in its turn, a necessary condition for the congruence of the forms $\mathbb{A}'$ and $\mathbb{B}'$.

Proof. $1 \Rightarrow 2$. Let us prove 1.

1. By virtue of condition (a), the letters $\alpha'_1, \ldots, \alpha'_n$ (as well as the letters $\alpha_1, \ldots, \alpha_n$) are pairwise distinct. Therefore the form $\mathbb{B}_2^*$ is obtained from the form $\mathbb{B}'_2$ by a substitution which, by virtue of condition (b), causes only a replacement of groups of free letters; namely, the groups of free letters whose representatives are $\alpha_1, \ldots, \alpha_n$ are changed, respectively, into the groups of free letters whose representatives are $\alpha'_1, \ldots, \alpha'_n$. Hence, since $\mathbb{A}_2 \simeq \mathbb{B}'_2$, we easily conclude that $\mathbb{A}_2$, $\mathbb{B}'_2$ and $\mathbb{B}_2^*$ are mutually variant forms. Likewise, from $\mathbb{A} \simeq \mathbb{B}$ we easily conclude that the forms $\mathbb{A}'$ and $\mathbb{B}'$ have one and the same number of symbols. We denote this number by l. Let f be the canonical mapping of the set of symbols constituting the form $\mathbb{A}'$ (we mean occurrences of symbols) onto the set of symbols constituting the form $\mathbb{B}'$. The parts $\mathbb{A}_2$ and $\mathbb{B}_2^*$ of the forms $\mathbb{A}'$ and $\mathbb{B}'$ (more precisely, the occurrences of the forms $\mathbb{A}_2$ and $\mathbb{B}_2^*$ obtained as the result of the replacements) are mutually corresponding with respect to f. Moreover, the elements of the groups of free letters of the part $\mathbb{A}_2$ of the form $\mathbb{A}'$, resp. of the part $\mathbb{B}_2^*$ of the form $\mathbb{B}'$, whose representatives are $\mathsf{a}_1, \ldots, \mathsf{a}_n$, resp. $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$, are bound letters of the form $\mathbb{A}'$, resp. $\mathbb{B}'$. The other free, resp. bound, letters of the parts $\mathbb{A}_2$ and $\mathbb{B}_2^*$ are free, resp. bound, letters of the forms $\mathbb{A}'$ and $\mathbb{B}'$. Now it is easy to see that for f condition (d) of the definition of congruence is satisfied (every letter located outside the parts $\mathbb{A}_2$ and $\mathbb{B}_2^*$ is bound or free according as the corresponding letter of $\mathbb{A}$ and $\mathbb{B}$ is bound or free; by virtue of the congruence of $\mathbb{A}_2$ and $\mathbb{B}'_2$, condition (d) for the parts $\mathbb{A}_2$ and $\mathbb{B}_2^*$, regarded as forms, is violated only with respect to the free occurrences of the letters $\mathsf{a}_1, \ldots, \mathsf{a}_n$ and $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$, but these free occurrences become bound occurrences upon passing to the forms $\mathbb{A}'$ and $\mathbb{B}'$). Therefore it suffices to show that to the groups of bound letters of the form $\mathbb{A}'$ there correspond, with respect to f, groups of bound letters of the form $\mathbb{B}'$ (since it will then follow that $\mathbb{B}'$ is obtained from $\mathbb{A}'$ by a simultaneous replacement of groups of bound letters). Obviously, the groups of bound letters of the form $\mathbb{A}'$, resp. $\mathbb{B}'$, lying entirely within the part $\mathbb{A}_2$, resp. $\mathbb{B}_2^*$, are "the same" as the groups of bound letters of the form $\mathbb{A}_2$, resp. $\mathbb{B}_2^*$ or (what is "the same") $\mathbb{B}'_2$, and consequently, by virtue of the mutual variance of the forms $\mathbb{A}_2$ and $\mathbb{B}_2^*$, to the groups of bound letters of the form $\mathbb{A}'$ lying entirely in $\mathbb{A}_2$ there correspond groups of bound letters from $\mathbb{B}'$. Consider now a group of bound letters $\mathcal{A}_i$ of the form $\mathbb{A}'$ with canonical representative a_i, $i \in \{1, 2, \ldots, n\}$ (more precisely, the canonical representative of which is the occurrence corresponding to the occurrence a_i). Further, let $\mathcal{B}_i$ be the group of bound letters of the form $\mathbb{B}'$ with canonical representative a'_i. It is easy to see that

$$\mathcal{A}_i = \mathcal{A}'_i \cup C_i \qquad \text{and} \qquad \mathcal{B}_i = \mathcal{B}'_i \cup D_i,$$

where $\mathcal{A}'_i$ (identifying an occurrence of a symbol in the form $\mathbb{A}'$ with the occurrence of the same symbol in the form $\mathbb{A}$ naturally corresponding to it) is the part, lying outside the part $\mathbb{A}_1$ of the form $\mathbb{A}$, of the group of bound letters whose canonical

representative is a_i, and C_i is the group of free letters of the part $\mathbb{A}_2$ whose canonical representative is the occurrence of the letter α_i in the part $\mathbb{A}_2$. The sets $\mathcal{B}'_i$ and D_i are defined analogously ($\mathbb{B}^*_2$ plays the role of $\mathbb{A}_2$). It is easy to see that the set $\mathcal{B}_i$ corresponds to the set $\mathcal{A}_i$ with respect to the mapping f. Now let E be a group of bound letters of the form $\mathbb{A}'$ whose canonical representative is none of the a_i, $i = 1, 2, \ldots, n$, and which does not lie entirely in the part $\mathbb{A}_2$ (in this case E lies entirely outside the part $\mathbb{A}_2$). Then E is the part, lying in $\mathbb{A}'$, of a group of bound letters E_1 of the form $\mathbb{A}$ located outside $\mathbb{A}_1$; moreover, if E'_1 is the group of bound letters of $\mathbb{B}$ corresponding to the group E_1, then E', the part (from $\mathbb{B}'$) corresponding to the part of E'_1 located outside $\mathbb{B}_1$, will be the group of bound letters of $\mathbb{B}'$ corresponding to the group E. There are no other cases.

Now, to prove the third item it suffices to note the following. Since $\mathbb{A}_2$ is free for $\mathbb{A}_1$ in the form $\mathbb{A}$ and every free letter of the part $\mathbb{A}_1$ that is a bound letter in $\mathbb{A}$ is a free letter of the form $\mathbb{A}_2$, the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ can also be defined as follows. The canonical representative a of a group of bound letters of the form $\mathbb{A}$ is a member of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ if and only if a acts on $\mathbb{A}_1$. Likewise, the sequence of canonical representatives associated with the forms $\mathbb{B}$, $\mathbb{B}^*_2$ and the part $\mathbb{B}_1$ of the form $\mathbb{B}$, and defined analogously to the original definition of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$, can be defined by a complete analogue of the last definition of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$. From this condition (a) easily follows. □

Remark. The requirement of item 3 that "every free letter of the part $\mathbb{A}_1$, resp. $\mathbb{B}_1$, which is bound in the form $\mathbb{A}$, resp. $\mathbb{B}$, be a free letter of the form $\mathbb{A}_2$, resp. $\mathbb{B}^*_2$" can be weakened, and instead of it one can require: "if a is a free letter of the form $\mathbb{A}_1$ which is bound in $\mathbb{A}$ and is not a free letter of the form $\mathbb{A}_2$, then the letter a' of $\mathbb{B}$ corresponding to the letter a is not a free letter of the form $\mathbb{B}^*_2$".

Theorem 2.3. *Let $\mathbb{A}, \mathbb{B}, \mathbb{A}_2, \mathbb{B}_2$ be forms such that $\mathbb{A} \simeq \mathbb{B}$ and $\mathbb{A}_2 \simeq \mathbb{B}_2$. Moreover, let $\mathbb{A}_1$ and $\mathbb{B}_1$ be mutually corresponding parts of the forms $\mathbb{A}$ and $\mathbb{B}$ whose type coincides with the type of the forms $\mathbb{A}_2$ and $\mathbb{B}_2$. Further, let $\mathsf{a}_1, \ldots, \mathsf{a}_n$ be such an enumeration of the canonical representatives of certain groups of bound letters of the form $\mathbb{A}$ that the canonical representative a of any group of bound letters of the form $\mathbb{A}$ is a member of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ if and only if a acts on the part $\mathbb{A}_1$ and a acts on the form $\mathbb{A}_2$ located in place of $\mathbb{A}_1$ (i.e. a, regarded as a letter, is a free letter of the form $\mathbb{A}_2$) and, moreover, the order among these representatives in the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ is the same as their order in the form $\mathbb{A}$. Furthermore, let $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$ be the occurrences of letters of $\mathbb{B}$ corresponding to the occurrences $\mathsf{a}_1, \ldots, \mathsf{a}_n$. We introduce the following notation: α'_i, resp. α_i, is the occurrence a'_i, resp. a_i, regarded as a letter, $i = 1, 2, \ldots, n$ (the letters $\alpha_1, \ldots, \alpha_n$ are pairwise distinct; the same can be said of the letters $\alpha'_1, \ldots, \alpha'_n$);*

$$\mathbb{B}^*_2 = (\alpha'_1, \ldots, \alpha'_n/\alpha_1, \ldots, \alpha_n)\mathbb{B}_2 = (\alpha'_1, \ldots, \alpha'_n \,/.\, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2,$$

where $(\alpha'_1, \ldots, \alpha'_n \,/.\, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2$ is the last member of the sequence considered in the definition of the value of the expression $(\alpha'_1, \ldots, \alpha'_n/\alpha_1, \ldots, \alpha_n)\mathbb{B}_2$; $\mathbb{A}' = (\mathbb{A}_2 \,//\, \mathbb{A}_1)\mathbb{A}$; $\mathbb{B}' = (\mathbb{B}^*_2 \,//\, \mathbb{B}_1)\mathbb{B}$. Then $\mathbb{A}' \simeq \mathbb{B}'$ if and only if (a') the sequence of canonical representatives

associated with the forms $\mathbb{B}$ *and* $\mathbb{B}_2^*$ *and the part* $\mathbb{B}_1$ *of the form* $\mathbb{B}$*, and defined in the same way as the sequence* $\mathsf{a}_1, \ldots, \mathsf{a}_n$*, coincides with the sequence* $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$*, and* (b′) *every free occurrence of any of the letters* $\alpha'_1, \ldots, \alpha'_n$ *in the form* $\mathbb{B}_2^* = (\alpha'_1, \ldots, \alpha'_n \,/_. \, \alpha_1, \ldots, \alpha_n)\mathbb{B}'_2$ *is a substituted occurrence.*

Proof. The necessity of the conditions (a′) and (b′) is easily verified. Let us prove sufficiency.

The proof of the sufficiency of the conditions (a′) and (b′) is based on Theorem 2.2. We have

$$\mathbb{A}' = (\mathbb{A}_2 \,//_. \, \mathbb{A}_1^*)\mathbb{A}^* \qquad \text{and} \qquad \mathbb{B}' = (\mathbb{B}_2^* \,//_. \, \mathbb{B}_1^*)\mathbb{B}^*, \tag{1}$$

where the right-hand sides of these graphic equalities are the last members of the sequences considered in the definition of the meaning of the expressions $(\mathbb{A}_2 \,//\, \mathbb{A}_1)\mathbb{A}$ and $(\mathbb{B}_2^* \,//\, \mathbb{B}_1)\mathbb{B}$. If we take into account these definitions, including the definition of the meaning of the expression $(\alpha'_1, \ldots, \alpha'_n/\alpha_1, \ldots, \alpha_n)\mathbb{B}_2$, and the remarks given with these definitions, we easily convince ourselves of the following. The form $\mathbb{B}'_2$ is a form congruent to the form $\mathbb{B}_2$ such that for each $i \in \{1, 2, \ldots, n\}$ the letter α'_i is free for α_i in $\mathbb{B}'_2$; $\mathbb{A} \simeq \mathbb{A}^*$; $\mathbb{B}^* \simeq \mathbb{B}$; the form $\mathbb{A}_2$ is free for $\mathbb{A}_1^*$ in $\mathbb{A}^*$; the form $\mathbb{B}_2^*$ is free for $\mathbb{B}_1^*$ in $\mathbb{B}^*$; $\mathbb{A}_1^* = \mathbb{A}_1$; $\mathbb{B}_1^* = \mathbb{B}_1$; and $\mathbb{A}_1$, $\mathbb{A}_1^*$, $\mathbb{B}_1$ and $\mathbb{B}_1^*$ are mutually corresponding parts of the mutually congruent forms $\mathbb{A}$, $\mathbb{A}^*$, $\mathbb{B}$ and $\mathbb{B}^*$; if we identify mutually corresponding occurrences of one and the same symbol of the mutually congruent forms $\mathbb{A}$ and $\mathbb{A}^*$, resp. $\mathbb{B}$ and $\mathbb{B}^*$, then we can say that the sequence of canonical representatives of groups of bound letters, defined analogously to the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$ in connection with the forms $\mathbb{A}^*$, $\mathbb{A}_2$ and the part $\mathbb{A}_1^*$ of the form $\mathbb{A}^*$, resp. in connection with the forms $\mathbb{B}^*$, $\mathbb{B}_2^*$ and the part $\mathbb{B}_1^*$ of the form $\mathbb{B}^*$, coincides with the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_n$, resp. $\mathsf{a}'_1, \ldots, \mathsf{a}'_n$.

Now it is easy to see that if in the role of the symbols

$$\mathbb{A},\ \mathbb{A}_1,\ \mathbb{A}_2,\ \mathbb{B},\ \mathbb{B}_1,\ \mathbb{B}_2,\ \mathbb{B}'_2$$

of Theorem 2.2 we take the symbols

$$\mathbb{A}^*,\ \mathbb{A}_1^*,\ \mathbb{A}_2,\ \mathbb{B}^*,\ \mathbb{B}_1^*,\ \mathbb{B}_2,\ \mathbb{B}'_2$$

(then in the role of $\mathbb{B}_2^*$ will appear $\mathbb{B}_2^*$, and in the role of the sequences

$$\mathsf{a}_1, \ldots, \mathsf{a}_n \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_n$$

will appear the sequences

$$\mathsf{a}_1, \ldots, \mathsf{a}_n \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_n,$$

since, for example, in the case of $\mathbb{A}^*$, $\mathbb{A}_1^*$, $\mathbb{A}_2$, if the canonical representative of a group of bound letters acts on the form $\mathbb{A}_2$ located in place of the part $\mathbb{A}_1^*$, then it necessarily acts also on the part $\mathbb{A}_1^*$), then the conditions (a) and (b) will be satisfied and, consequently, by virtue of the graphic equalities (1) and Theorem 2.2, we shall have $\mathbb{A}' \simeq \mathbb{B}'$. □

Theorem 2.4. *Let $\mathbb{A}, \mathbb{B}, \mathbb{A}_2, \mathbb{B}_2$ be forms such that $\mathbb{A} \simeq \mathbb{B}$ and $\mathbb{A}_2 \simeq \mathbb{B}_2$. Further, let $\mathbb{A}_1$ and $\mathbb{B}_1$ be mutually corresponding parts of the forms $\mathbb{A}$ and $\mathbb{B}$ whose type coincides with the type of the forms $\mathbb{A}_2$ and $\mathbb{B}_2$.*

1. *If the form $\mathbb{A}_2$ is completely free for $\mathbb{A}_1$ in $\mathbb{A}$, and $\mathbb{B}_2$ is completely free for $\mathbb{B}_1$ in $\mathbb{B}$, then*

$$(\mathbb{A}_2 \mathbin{//} \mathbb{A}_1)\mathbb{A} = (\mathbb{A}_2 \mathbin{//:} \mathbb{A}_1)\mathbb{A} \simeq (\mathbb{B}_2 \mathbin{//:} \mathbb{B}_1)\mathbb{B} = (\mathbb{B}_2 \mathbin{//} \mathbb{B}_1)\mathbb{B}. \tag{2}$$

2. *If the form $\mathbb{A}_2$ is free for $\mathbb{A}_1$ in $\mathbb{A}$, and $\mathbb{B}_2$ is free for $\mathbb{B}_1$ in $\mathbb{B}$ and, moreover, $\mathbb{A}_1 \simeq \mathbb{B}_1$, then (2) holds.*

3. *If $\mathbb{A}_1 \simeq \mathbb{B}_1$, then $(\mathbb{A}_2 \mathbin{//} \mathbb{A}_1)\mathbb{A} \simeq (\mathbb{B}_2 \mathbin{//} \mathbb{B}_1)\mathbb{B}$.*

Proof. The validity of the graphic equalities (in the case of items 1 and 2) is obvious. Therefore it suffices to prove that

$$(\mathbb{A}_2 \mathbin{//} \mathbb{A}_1)\mathbb{A} \simeq (\mathbb{B}_2 \mathbin{//} \mathbb{B}_1)\mathbb{B}.$$

The validity of this congruence follows from the preceding theorem, since in our case, as is easily seen, the symbols defined in the same way as in the formulation of the preceding theorem satisfy the conditions (a′) and (b′) (in the case of items 2 and 3 this follows easily from the congruence $\mathbb{A}_1 \simeq \mathbb{B}_1$, and in the case of item 1 from the obvious fact that the indicated sequences are empty). □

Remark. Item 1 of Theorem 2.1 can be regarded as a corollary of Theorem 2.4 (the proof of Theorem 2.4 does not depend on Theorem 2.1). Namely, item 1 of Theorem 2.1 is derived from Theorem 2.4 as follows. Without loss of generality one may assume that a_i, $\mathbb{A}_i$ and $\mathbb{B}_i$ are forms of one and the same type. Let

$$\mathbb{A}' = (\mathbb{A}_1, \ldots, \mathbb{A}_n \mathbin{/.} \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A}^* \quad \text{and} \quad \mathbb{B}' = (\mathbb{B}_1, \ldots, \mathbb{B}_n \mathbin{/.} \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{B}^*,$$

where the right-hand sides of these graphic equalities are the last members of the sequences considered in the definition of the meaning of the expressions

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A} \quad \text{and} \quad (\mathbb{B}_1, \ldots, \mathbb{B}_n/\mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{B}.$$

Then obviously $\mathbb{A}^* \simeq \mathbb{B}^*$ and for each $i \in \{1, 2, \ldots, n\}$ the form $\mathbb{A}_i$ is (completely) free for a_i in $\mathbb{A}^*$, and the form $\mathbb{B}_i$ is free for a_i in $\mathbb{B}^*$. Therefore the required congruence

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n \mathbin{/.} \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{A}^* \simeq (\mathbb{B}_1, \ldots, \mathbb{B}_n \mathbin{/.} \mathsf{a}_1, \ldots, \mathsf{a}_n)\mathbb{B}^*$$

is obtained by several applications of item 1 of the preceding theorem (a simultaneous substitution can be carried out as a sequence of simple substitutions; one has to mark in advance all the letters being replaced, since as a result of each simple replacement the condition of "complete freedom" is preserved with respect to the letters being replaced).

Theorem 2.5. *By replacing a part $\mathbb{A}_1$ of a form $\mathbb{A}$ by a form $\mathbb{A}_2$ congruent to the part $\mathbb{A}_1$, one obtains a form $\mathbb{A}'$ congruent to the form $\mathbb{A}$. Moreover, in order to obtain the form $\mathbb{A}'$ from the form $\mathbb{A}$ it suffices to replace several groups of bound letters located in the part $\mathbb{A}_1$.*

Indeed, it suffices to assume that $\mathbb{B} = \mathbb{A}$, $\mathbb{B}_1 = \mathbb{A}_1$, $\mathbb{B}_2 = \mathbb{A}_1$ and to apply item 2 of Theorem 2.4 (the form $\mathbb{A}_2$ is free for the part $\mathbb{A}_1$ in $\mathbb{A}$).

Corollary. By replacing non-overlapping parts $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of a form $\mathbb{A}$, respectively, by forms $\mathbb{B}_1, \ldots, \mathbb{B}_n$ congruent to them, one obtains a form $\mathbb{A}'$ congruent to the form $\mathbb{A}$.

Finally, let us note that the following is easily proved.

Theorem 2.6. *Any corresponding parts $\mathbb{A}_1$ and $\mathbb{B}_1$ of mutually congruent forms $\mathbb{A}$ and $\mathbb{B}$ are mutually variant forms. Moreover, any group of free letters of the part $\mathbb{A}_1$ is either a proper part of a group of bound letters of the form $\mathbb{A}$ located in the part $\mathbb{A}_1$, or a part of a group of free letters of the form $\mathbb{A}$ located in the part $\mathbb{A}_1$.*

2.5 Further Remarks on Contracting Symbols

2.5.1

From the definition of a contracting symbol (i.e. of a derived symbol) the type and the weight of this derived symbol are naturally induced and, if it is an operator sign, then the rule is also induced according to which the corresponding derived compound operators (quantifiers) are to be formed. Likewise, from the definition of a derived logical-special operator sign its logicality indicator is naturally induced, and from the definition of a derived partial operator sign its binding indicator is induced.

If we add contracting symbols to the alphabet of the theory $\mathscr{T}$, resp. $\mathscr{T}'$, resp. $\mathscr{T}''$, as symbols of the corresponding types (i.e. if we take each contracting symbol as a basic symbol of the same type), we obtain the so-called **theory extended by contracting symbols**, which we shall denote by $\widetilde{\mathscr{T}}$, resp. $\widetilde{\mathscr{T}}'$, resp. $\widetilde{\mathscr{T}}''$. It is assumed that in the case of $\widetilde{\mathscr{T}}'$ the labeled operators among the derived operators are the derived two-place propositional connectives, the derived two-place predicate constants, the derived two-place functional constants, and only they.

Obviously, no contracting symbol enters into the alphabetic enumeration of the symbols of the extended theory under consideration.

We shall deal with such definitions of contracting symbols that the class of (compound) operators of the theory $\widetilde{\mathscr{T}}$, resp. $\widetilde{\mathscr{T}}'$, resp. $\widetilde{\mathscr{T}}''$, will be the union of the class of compound operators of the theory $\mathscr{T}$ and the class of compound derived operators. Moreover, those general requirements will be satisfied that we imposed on operator signs, on the rules by means of which compound operators are obtained, and on the class of compound operators.

If the set of contracting symbols is countable, then $\widetilde{\mathscr{T}}$ will be a $\mathscr{T}$-theory. Otherwise $\widetilde{\mathscr{T}}$ will be a generalized $\mathscr{T}$-theory. Consequently, also in the latter case all the basic results obtained in the theory $\mathscr{T}$ remain valid for the theory $\widetilde{\mathscr{T}}$. Sometimes the system of derived symbols is not fixed and new derived symbols are introduced

gradually. In this case as well the results obtained in the theory $\mathscr{T}$ are in the main applicable to the theory $\widetilde{\mathscr{T}}$.

Below we shall use the forms of the theory $\widetilde{\mathscr{T}}$ to denote forms of the theory $\mathscr{T}$ of the same type, in such a way that each form of the theory $\widetilde{\mathscr{T}}$ denotes at least one form of the theory $\mathscr{T}$ (in cases important for practice each form of the theory $\widetilde{\mathscr{T}}$ denotes a form determined uniquely up to congruence, and in a number of important cases a uniquely determined form). On the basis of this circumstance the axioms and rules of inference of the theory $\widetilde{\mathscr{T}}$ are naturally induced from the axioms and rules of inference of the theory $\mathscr{T}$. Namely, as follows: a formula $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ is accepted as an axiom if and only if it denotes at least one axiom of the theory $\mathscr{T}$. Likewise, a formula $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ is an immediate consequence of the formulas $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of the theory $\widetilde{\mathscr{T}}$ if and only if there are formulas $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathbb{A}'$ of the theory $\mathscr{T}$, denoted respectively by the formulas $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$, such that $\mathbb{A}'$ is an immediate consequence of the formulas $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$.

Below we shall assume that in an arbitrary theory $\mathscr{T}_a$ which is an extension of the given theory $\mathscr{T}$, and whose forms denote forms of the same type of the theory $\mathscr{T}$, the axioms and rules of inference are induced in the above-mentioned way.

In general, a generalized theory $\mathscr{T}_a$ we shall call a theory **obtained by an extension of the theory** $\mathscr{T}$, or briefly an **extended theory of the theory** $\mathscr{T}$, or, still shorter, an **extended theory**, if the alphabet of the theory $\mathscr{T}_a$ is obtained from the alphabet of the theory $\mathscr{T}$ by adding operator signs, operators and constants of arbitrary types, without changing the class of quantifier letters (it is not excluded that among the additional symbols there occur partial operator signs, logical-special operator signs and logical-special operators). These last symbols we call the **additional symbols of the extended theory**. The **extension of a given generalized theory** is defined similarly.

If $\mathbb{A}$ is a metavariable whose domain of values is the class of all terms, resp. formulas, of the theory $\mathscr{T}$, and $\mathscr{T}_a$ is an extended theory, then the class of all terms, resp. formulas, of the theory $\mathscr{T}_a$ we shall call the **domain of values in** $\mathscr{T}_a$ of the metavariable $\mathbb{A}$. An arbitrary element of this domain we shall call a **value from** $\mathscr{T}_a$ (**relative to** $\mathscr{T}_a$) of the metavariable $\mathbb{A}$. Accordingly, the meaning of the term **system of values from** $\mathscr{T}_a$ (**relative to** $\mathscr{T}_a$) of the metavariables $\mathbb{B}_1, \ldots, \mathbb{B}_n$ is to be understood (if the domain of values of the metavariable $\mathbb{B}_j$ is neither the class of all terms of the theory $\mathscr{T}$ nor the class of all formulas of the theory $\mathscr{T}$, then the values of the variable $\mathbb{B}_j$ should be taken from its domain of values).

2.5.2

To write the definitions of contracting symbols we shall use the symbol "———" ("long dash"), which is read: "... represents the contracted version of ...". Here, in order to obtain the definition of a contracting symbol, one must put at the place of the first ellipsis (i.e. to the left of the symbol "———") a metamathematical expression composed of the symbol being defined, previously introduced contracting

symbols, symbols from the alphabet of the given theory, and metavariables; and at the place of the second ellipsis one must put a metamathematical expression composed of previously introduced contracting symbols, symbols from the alphabet of the given theory, metavariables, and additional symbols of the theory $\widetilde{\mathscr{T}}_a$, which upon substitution in it, instead of the metavariables, of systems of their values relative to the theory $\widetilde{\mathscr{T}}$, turns into a uniquely determined form of the theory $\widetilde{\mathscr{T}}_a$. Here $\widetilde{\mathscr{T}}_a$ is a theory obtained by an extension of the theory $\widetilde{\mathscr{T}}$, each form of which is a denotation of a uniquely determined form of the same type of the theory $\widetilde{\mathscr{T}}$. Moreover, the type of the indicated form of the theory $\widetilde{\mathscr{T}}_a$ does not depend on the choice of the system of values of the mentioned metavariables relative to $\widetilde{\mathscr{T}}$. Further, there is an algorithm which makes it possible, for any form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}_a$, to find a form of the theory $\widetilde{\mathscr{T}}$ of which the form $\mathbb{A}$ is a denotation (the case is not excluded when $\widetilde{\mathscr{T}}_a$ and $\widetilde{\mathscr{T}}$ coincide with each other; in this case it is understood that the form $\mathbb{A}$ denotes itself). The metamathematical expressions under consideration are called, respectively, the **left-hand side** and the **right-hand side of the definition**. The new contracting symbol is to be imagined as a derived symbol of such a type that for an arbitrary system of values of the metavariables relative to $\widetilde{\mathscr{T}}$ in the left-hand side of the definition a form of the theory $\widetilde{\mathscr{T}}$ is obtained whose type coincides with the type of the form obtained in the right-hand side of the definition for the same system of values of the metavariables (we mean all the metavariables occurring in the definition). This condition, on the basis of the indicated naturalness, should uniquely determine the type of the new derived symbol and, in the case when it is an operator sign, should naturally and uniquely determine the class of compound operators corresponding to this operator sign. These expressions can be written with the help of some notations (for example, in abbreviated form). In such cases, in these notations one must always imagine the corresponding expression. Finally, as the **basic meaning of the definition** one should regard the following: the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, obtained in the left-hand side of the definition upon substitution of any admissible system of values of the metavariables relative to $\widetilde{\mathscr{T}}$, is a contracted version (contracted notation) of the form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$, a denotation of which is the form $\mathbb{A}_a$ of the theory $\widetilde{\mathscr{T}}_a$ obtained in the right-hand side of the definition upon substituting the same system of values of the metavariables. For simplicity one may say that $\mathbb{A}_1$ is a contracted version (contracted notation) of the form $\mathbb{A}_a$ (understanding by the form $\mathbb{A}_a$ the form $\mathbb{A}$). Let us note here that if in the definition of a contracting symbol the definition of the phrase "admissible system of values of the metavariables relative to $\widetilde{\mathscr{T}}_\beta$", where $\widetilde{\mathscr{T}}_\beta$ is an extension of the theory $\widetilde{\mathscr{T}}$, is not specially given, then it is understood that the meaning of this phrase is the same as the meaning of the phrase "system of values of the metavariables relative to the theory $\widetilde{\mathscr{T}}_\beta$".

For an example let us consider the system of contracting symbols applied in the

theory $\mathcal{P}'_1$ and introduced by the following definitions:

D1. $t \text{ ——— } f \longrightarrow f.$
D2. $[\neg\mathbb{A}] \text{ ——— } \mathbb{A} \longrightarrow f.$
D3. $[\mathbb{A} \nleftarrow \mathbb{B}] \text{ ——— } \neg[\mathbb{B} \longrightarrow \mathbb{A}].$
D4. $[\mathbb{A} \vee \mathbb{B}] \text{ ——— } [\mathbb{A} \longrightarrow \mathbb{B}] \longrightarrow \mathbb{B}.$
D5. $[\mathbb{A} \wedge \mathbb{B}] \text{ ——— } [\mathbb{A} \nleftarrow \mathbb{B}] \nleftarrow \mathbb{B}.$
D6. $[\mathbb{A} \longleftrightarrow \mathbb{B}] \text{ ——— } [\mathbb{A} \longrightarrow \mathbb{B}] \wedge [\mathbb{B} \longrightarrow \mathbb{A}].$

D7. $[\mathbb{A} \nleftrightarrow \mathbb{B}] \text{ ——— } [\mathbb{A} \nleftarrow \mathbb{B}] \vee [\mathbb{B} \nleftarrow \mathbb{A}].$
D8. $[\mathbb{A} \leftarrow \mathbb{B}] \text{ ——— } \mathbb{B} \longrightarrow \mathbb{A}.$
D9. $[\mathbb{A} \nrightarrow \mathbb{B}] \text{ ——— } \mathbb{B} \nleftarrow \mathbb{A}.$
D10. $[\mathbb{A} \mathbin{\overline{\vee}} \mathbb{B}] \text{ ——— } \neg\mathbb{A} \wedge \neg\mathbb{B}.$
D11. $[\mathbb{A} \mathbin{\overline{\wedge}} \mathbb{B}] \text{ ——— } \neg\mathbb{A} \vee \neg\mathbb{B}.$

Remark. By a single application of the contracting symbol t introduced by definition D1 one can write in contracted form only the single formula $[f \longrightarrow f]$. By a single application of each of the contracting symbols introduced by definitions D2–D11 one can write in contracted form all the formulas of a completely definite infinite class of formulas. Therefore some authors call each of the definitions D2–D11 a **definition-scheme**. Since each definition Di $(i = 1, 2, \ldots, 11)$ introduces a single contracting symbol, we consider that each Di is a definition and not a definition-scheme. Let us note here that the term "definition-scheme" will be given below a more natural content.

By a careful examination of the definitions D1–D11 we convince ourselves that the derived symbols introduced by these definitions are derived logical connectives (derived relational logical operators). Moreover, the derived logical connective "t" introduced by definition D1 is nullary (i.e. it is a derived propositional constant), the derived logical connective "$\neg$" is one-place, and the remaining ones are two-place.

Let us note that every derived operator introduced by the definitions D1–D11 has an "operator character". This means that if $\mathbb{A}$ and $\mathbb{B}$ are arbitrary forms (formulas) of the theory $\mathcal{P}'_1$ and σ is an arbitrary two-place derived operator among the operators introduced by the definitions D3–D11, then each of the forms $[\mathbb{A}\,\sigma\,\mathbb{B}]$, $[\neg\mathbb{A}]$ and t of the theory $\widetilde{\mathcal{P}}'_1$ is the contracted form of a uniquely determined form of the theory $\mathcal{P}'_1$ (in what follows we shall weaken here the requirement of uniqueness, demanding uniqueness only up to congruence).

Sometimes such definitions are used which introduce derived operators that do not have an "operator character". An example of such a definition is

$$[\sigma\,\mathbb{A}_1\mathbb{A}_2] = \mathbb{A}_3 \text{ ——— } \sigma_1\,\mathbb{A}_1\mathbb{A}_2\mathbb{A}_3, \tag{1}$$

where $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ are metavariables whose domain of values is the class of all terms of the given theory, σ_1 is a basic ternary special relational operator, and σ is the contracting symbol introduced by definition (1) (obviously σ is a derived special substantive two-place operator; if T_1 and T_2 are terms of the given theory, then $\sigma T_1 T_2$ is not a contracted form of a form of the given theory). The establishment of general properties of such definitions is connected with considerable difficulties. Moreover, they are rarely applied. Therefore we shall in the main consider only such definitions as introduce contracting symbols having an "operator character".

We shall in the main study such definitions as have the following form:

$$\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}, \tag{2}$$

where σ is the contracting symbol introduced by this definition and $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ are metavariables; moreover, either (I) the domain of values of each metavariable $\mathbb{A}_i$ is the class of all formulas (of the given theory), or (II) the domain of values of each metavariable $\mathbb{A}_i$ is the class of all terms, or (III) the domain of values of some (at least one, but not all) of the metavariables $\mathbb{A}_i$ is the class of all formulas, and for the remaining $\mathbb{A}_i$ it is the class of all terms; moreover, the domain of values of each metavariable a_i is such a class K_i of quantifier letters that contains all the variables whose type coincides with the type of some of its elements, and all the quantifier constants whose type coincides with the type of some of its elements that is a constant ($i = 1, 2, \ldots, m$). Here the natural numbers m and n may also take zero values, subject to the following condition: $m \neq 0 \Rightarrow n \neq 0$. If $m \neq 0$, then it is natural to consider that σ is a **derived operator sign with weight** (n, m). Otherwise one should consider that σ is a **derived simple operator with weight** n. Moreover, if $m = n = 0$, then σ will be a derived propositional constant or a derived object constant, depending on whether $\mathbb{B}$ is a formula or a term. Besides, it is obvious that the derived symbol σ is logical in case (I), special in case (II), and logical-special in case (III). It is relational or substantive depending on whether the right-hand side $\mathbb{B}$ of the definition (a form of the theory $\widetilde{\mathscr{T}}$) is a formula or a term (for all systems of values of its metavariables).

Below, where the contrary does not follow from the context, we shall assume that the definition of a contracting symbol has the form (2) subject to the above general conditions (requirements), and we shall use the terminology introduced in the preceding paragraph (it is assumed that $\mathbb{B}$ is a metamathematical expression which turns into a form of the theory $\widetilde{\mathscr{T}}$ or into the denotation of a uniquely determined form of the theory $\widetilde{\mathscr{T}}$ upon substitution of any system of values of its metavariables relative to the theory $\widetilde{\mathscr{T}}$). Obviously, the derived operators introduced by such definitions will have an "operator character". These conditions are satisfied for the definitions D1–D11 (the right-hand side of each of these definitions turns into a formula of the theory $\widetilde{\mathcal{P}}'_1$ upon substitution of any system of values of the metavariables occurring in it relative to the theory $\widetilde{\mathcal{P}}'_1$).

Chapter 3

Some Notions Needed for Introducing the Various Types of Definitions of Contracting Symbols

3.1 Forms Constructed Using Metavariables and Substitutions

3.1.1

In this section we define and investigate certain notions that in the next section are used for introducing the basic types of definitions.

Let $\sigma_1, \ldots, \sigma_k$ be metavariables such that for each $i \in \{1, 2, \ldots, k\}$ the domain of values of the metavariable σ_i is some class K_i of words of the given theory. If K_i is some class of letters, resp. of quantifier letters (it may contain letters of different types), then we call σ_i a **metaletter**, resp. a **quantifier metaletter**.

We introduce the notions of **formula** and **term constructed using the metavariables** $\sigma_1, \ldots, \sigma_k$ by a recursive definition obtained from the definition of the notions of formula and term as follows: (a) in the recursive definition of formulas and terms the words "formula" and "term" are replaced by the terms "formula constructed using the metavariables $\sigma_1, \ldots, \sigma_k$" and "term constructed using the metavariables $\sigma_1, \ldots, \sigma_k$"; (b) to the direct clauses of the recursive definition one adds, for each $i \in \{1, 2, \ldots, k\}$, zero or several clauses. Namely, (1) if K_i is some class of formulas, resp. terms, then one adds the clause: "σ_i is a formula constructed, resp. a term constructed, using the metavariables $\sigma_1, \ldots, \sigma_k$"; (2) if K_i is some class of letters of a definite type (we mean only the types introduced on page 12 of the original), then one adds a clause in which it is stated that σ_i is a letter of the corresponding type (if K_i is some class of propositional, resp. object, letters, then there is no need to add a clause corresponding to case (1)). In addition, one has to add the following clause: "If σ is an n-place m-ary operator sign, $\mathsf{a}_1, \ldots, \mathsf{a}_m$ is a sequence of symbols each member of which is a letter or a metaletter, $\mathsf{a}_{i_1}, \ldots, \mathsf{a}_{i_l}$ is the sequence of metaletters obtained

from the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_m$ by deleting the (mathematical) letters, and, moreover, for each system of values of the metaletters $\mathsf{a}_{i_1}, \ldots, \mathsf{a}_{i_l}$ the word $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m$ is an operator (quantifier), then by definition $\sigma\mathsf{a}_1 \cdots \mathsf{a}_m$ is an n-place operator (quantifier) whose type coincides with the type of the operator sign σ."

Obviously, a formula constructed (resp. a term constructed) using the metavariables $\sigma_1, \ldots, \sigma_k$ turns, upon substitution in it of any system of values of these variables, into a formula (resp. a term) of the given theory. It is easy to see that this property is characteristic of formulas, resp. terms, constructed using the metavariables $\sigma_1, \ldots, \sigma_k$, within the class of all words of the extended alphabet obtained from the alphabet of the given theory by adding the symbols $\sigma_1, \ldots, \sigma_k$.

Remark. To introduce the types of definitions of contracting symbols it suffices to introduce the notions of "formulas and terms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$" in the case when for each $i \in \{1, 2, \ldots, k\}$ the following condition holds: the domain of values of the metavariable σ_i is (1) the class of formulas (of the given theory), or (2) the class of terms, or (3) a non-empty class of quantifier letters which together with any variable, resp. constant, contains all the variables of the same type, resp. contains all the quantifier letters of the same type.

The general name for formulas and terms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ will, by definition, be **form constructed using the metavariables** $\sigma_1, \ldots, \sigma_k$. Since the forms, formulas and terms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ are denotations of forms, formulas and terms of the given theory, we shall often call them simply forms, formulas and terms, respectively.

If $\alpha_1, \ldots, \alpha_n$ are operators of the given theory and $\mathbb{A}$ is a form constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ in which no operators of the given theory other than $\alpha_1, \ldots, \alpha_n$ occur, then we shall call $\mathbb{A}$ a **form constructed using the metavariables** $\sigma_1, \ldots, \sigma_k$ **and the operators** $\alpha_1, \ldots, \alpha_n$. The corresponding notions for formulas and terms are induced from this.

3.1.2

Now we introduce the notions of **formula and term constructed using the metavariables** $\sigma_1, \ldots, \sigma_k$ **and substitutions of type** $(—/—)$ **relative to propositional and object letters** (and hence the notion of form) by a recursive definition obtained from the definition of the notions of formula and term constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ if: (a) in the recursive definition of formulas and terms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ the terms "formula constructed using the metavariables $\sigma_1, \ldots, \sigma_k$" and "term constructed using the metavariables $\sigma_1, \ldots, \sigma_k$" are replaced by the terms "formula constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ and substitutions of type $(—/—)$ relative to propositional and object letters" and "term constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ and substitutions of type $(—/—)$ relative to propositional and object letters"; (b) to the direct clauses of the recursive definition one adds the following clause: "If $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ and substitutions of type $(—/—)$ relative to propositional and object letters, and for each $i \in \{1, 2, \ldots, n\}$ X_i is a letter from the alphabetic

enumeration of propositional and object letters of the theory $\mathscr{T}$ or a metaletter each value of which is a member of the alphabetic enumeration of propositional letters of the theory $\mathscr{T}$ or a metaletter each value of which is a member of the alphabetic enumeration of object letters of the theory $\mathscr{T}$, so that $\mathbb{A}_i$ and X_i are forms constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ and substitutions of type (—/—) relative to propositional and object letters, of one and the same type, then $(\mathbb{A}_1, \ldots, \mathbb{A}_n/X_1, \ldots, X_n)\mathbb{A}$ is a form constructed using the metavariables $\sigma_1, \ldots, \sigma_k$ and substitutions of type (—/—) relative to propositional and object letters, whose type coincides with the type of $\mathbb{A}$." Finally, just as in the preceding paragraph, one can introduce the following notion: a **form constructed** (resp. **formula constructed**, resp. **term constructed**) **using the metavariables** $\sigma_1, \ldots, \sigma_k$, **the operators** $\alpha_1, \ldots, \alpha_n$ **and substitutions of type** (—/—) **relative to propositional and object letters**.

3.2 Substitution Quantifiers and Direct Substitution Quantifiers

3.2.1

We shall assume that a theory $\mathscr{T}$ and several of its extensions are being considered, whose forms denote forms of the theory $\mathscr{T}$ *directly or indirectly*. For example, if the forms of a theory $\mathscr{T}_2$ directly denote forms of a theory $\mathscr{T}_1$, and the forms of $\mathscr{T}_1$ directly denote forms of $\mathscr{T}$, then it must be assumed that the forms of $\mathscr{T}_2$ ultimately denote forms of $\mathscr{T}$ (indirectly). We denote the class of the indicated extensions by $\mathscr{A}$. We shall assume that the class $\mathscr{A}$ contains $\mathscr{T}$ ($\mathscr{T}$ is an **improper extension** of the theory $\mathscr{T}$). It is assumed that every form of any theory taken from $\mathscr{A}$ denotes at least one form of the theory $\mathscr{T}$. Let us suppose also that the alphabet of any theory from $\mathscr{A}$ contains neither the symbol S nor the symbol R, with or without indices.

To any theory $\mathscr{T}_\beta$ that contains no symbols S and R, with or without indices, and is an extension of $\mathscr{T}$, we associate the so-called **quasi-extended theory** $\mathscr{T}_{\beta\mathrm{S}}$, resp. $\mathscr{T}_{\beta\mathrm{R}}$, resp. $\mathscr{T}_{\beta SR}$, whose alphabet is obtained from the alphabet of $\mathscr{T}_\beta$ by adding the symbols S^{n+1} ($n = 1, 2, \ldots$), resp. R^{n+1} ($n = 1, 2, \ldots$), resp. S^{n+1} ($n = 1, 2, \ldots$) and R^{n+1} ($n = 1, 2, \ldots$), as so-called **complex operator signs**. The symbol S^{n+1} is called the **sign of** n**-place substitution**, and R^{n+1} the **sign of** n**-place direct substitution**.

Remark. We need to consider only some examples of quasi-extended theories. Therefore there is no need to work out a precisely defined suitable notion. The term "quasi-extended theory" indicates only that the object denoted by this term is analogous to an extended theory but is nevertheless not an extended theory.

The formal characteristics of the operator signs S^{n+1} and R^{n+1} are introduced as follows. The quantifiers corresponding to the operator sign S^{n+1}, resp. R^{n+1}, called **substitution quantifiers**, resp. **direct substitution quantifiers**, are those and only those words which have the form $\mathrm{S}^{n+1}X_1 \cdots X_n$, resp. $\mathrm{R}^{n+1}X_1 \cdots X_n$, where each X_i is a propositional letter or an object letter from the alphabetic enumeration of letters; the weight of the quantifier $\mathrm{S}^{n+1}X_1 \cdots X_n$, resp. $\mathrm{R}^{n+1}X_1 \cdots X_n$, is $n + 1$, and

for it, as the i-th operand, $i \leq n$, those and only those forms are admissible whose type coincides with the type of the form X_i, while as the $(n+1)$-th operand any form is admissible; the type of a form of the shape $\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, resp. $\mathrm{R}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, coincides with the type of the form $\mathbb{A}$; the letters bound by the operator sign S^{n+1}, resp. R^{n+1}, in the form $\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, resp. $\mathrm{R}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, are considered (by definition) to be the occurrences of the operator letters $X_1, \ldots, X_n$ in the quantifier $\mathrm{S}^{n+1}X_1 \cdots X_n$, resp. $\mathrm{R}^{n+1}X_1 \cdots X_n$, and the free occurrences of these operator letters in the operand (in the part) $\mathbb{A}$.

Thus the quantifiers $\mathrm{S}^{n+1}X_1 \cdots X_n$ and $\mathrm{R}^{n+1}X_1 \cdots X_n$ bind letters only in the last operand.

For simplicity, in the quantifiers under consideration we shall omit the upper indices if this causes no misunderstanding.

Below we shall assume that

$$(\mathbb{A}_1, \ldots, \mathbb{A}_n \,/_{a\mathrm{S}}\, X_1, \ldots, X_n)\mathbb{A}, \quad \text{resp.} \quad (\mathbb{A}_1, \ldots, \mathbb{A}_n \,/_{a\mathrm{R}}\, X_1, \ldots, X_n)\mathbb{A},$$

is a simplified notation for the form

$$\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}, \quad \text{resp.} \quad \mathrm{R}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}.$$

Moreover, any form of the theory $\mathscr{T}_{a\mathrm{R}}$, resp. $\mathscr{T}_{a\mathrm{SR}}$, or any part of a form of the theory $\mathscr{T}_{a\mathrm{R}}$, resp. $\mathscr{T}_{a\mathrm{SR}}$, of the shape

$$\mathrm{R}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A},$$

where $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms containing no direct substitution quantifiers, will be regarded as a denotation of that form of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$, which is the result of the substitution of type $(—/—)$, in the form $\mathbb{A}$ of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$, of the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$, in place of the letters $X_1, \ldots, X_n$. On the basis of this agreement any form $\mathbb{B}_1$ of the theory $\mathscr{T}_{a\mathrm{R}}$, resp. $\mathscr{T}_{a\mathrm{SR}}$, will be regarded as a denotation of a uniquely determined form $\mathbb{B}$ of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$, and as a form of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$. The reconstruction of the form $\mathbb{B}$ from the form $\mathbb{B}_1$ is carried out by the following algorithm. We find in $\mathbb{B}_1$ the first occurrence of a direct substitution quantifier in whose scope there is no other direct substitution quantifier, carry out the corresponding substitution, and replace by the result of this substitution the part of the form $\mathbb{B}_1$ corresponding to the mentioned direct substitution quantifier; the form $\mathbb{B}_{11}$ obtained in this way we transform by the same method into $\mathbb{B}_{12}$, and so on. (This process ends with the obtaining of a form of the theory $\mathscr{T}_a$, resp. $\mathscr{T}_{a\mathrm{S}}$. The result will be the same if at every step, instead of the first occurrence, we take any direct substitution quantifier with a minimal scope.) Any form of the theory $\mathscr{T}_{a\mathrm{S}}$, or any part of a form of the theory $\mathscr{T}_{a\mathrm{S}}$, of the shape $\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, where $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms containing no substitution quantifiers, will be regarded as a denotation of that form $\mathbb{C}$ of the theory $\mathscr{T}$ which is the result of the substitution of type $(—/—)$, in the form $\mathbb{A}'$, of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ in place of the letters $X_1, \ldots, X_n$, where $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathbb{A}'$

are such forms of the theory $\mathscr{T}$ of which the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are denotations (obviously, if the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathbb{A}'$ are determined uniquely, then the form $\mathbb{C}$ is determined uniquely as well). On the basis of this agreement any form $\mathbb{B}_1$ of the theory $\mathscr{T}_{a\mathrm{S}}$ will be regarded as a denotation of a form $\mathbb{B}$ of the theory $\mathscr{T}$ (determined uniquely or non-uniquely) and as a form of the theory $\mathscr{T}$. The reconstruction of the form $\mathbb{B}$ from the form $\mathbb{B}_1$ is carried out by a conditional algorithm analogous to the algorithm considered above (we require an algorithm by which for any form $\mathbb{A}_a$ of the theory $\mathscr{T}_a$ one can find a form $\mathbb{A}$ of the theory $\mathscr{T}$ of which $\mathbb{A}_a$ is a denotation). Any form of the theory $\mathscr{T}_{a\mathrm{SR}}$, or any such part of a form of the theory $\mathscr{T}_{a\mathrm{SR}}$ not lying in the scope of a direct substitution quantifier, which has the shape $\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$, where $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms containing no substitution quantifiers of the shape $\mathrm{S}^{k+1}Y_1 \cdots Y_k$, will be regarded as a denotation of that form $\mathbb{C}$ of the theory $\mathscr{T}$ which is the result of the substitution of type (—/—), in the form $\mathbb{A}'$, of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ in place of the letters $X_1, \ldots, X_n$, where $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathbb{A}'$ are such forms of the theory $\mathscr{T}$ of which the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are denotations (the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms of the theory $\mathscr{T}_{a\mathrm{R}}$; therefore in accordance with the above they denote forms of the theory $\mathscr{T}_a$, and the latter denote forms of the theory $\mathscr{T}$, so that the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ ultimately denote forms of the theory $\mathscr{T}$). In spite of what has been said, if $\mathrm{S}^{n+1}X_1 \cdots X_n\mathbb{A}_1 \cdots \mathbb{A}_n\mathbb{A}$ is a part of a form $\mathbb{B}$ of the theory $\mathscr{T}_{a\mathrm{SR}}$ lying in the scope of a direct substitution quantifier, and $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are forms containing no substitution quantifiers of the shape $\mathrm{S}^{k+1}Y_1 \cdots Y_k$, then this part cannot in general be replaced by a form $\mathbb{C}'$ defined analogously to the form $\mathbb{C}$ considered above (such a replacement may cause a replacement of the forms of the theories $\mathscr{T}_{a\mathrm{S}}$ and $\mathscr{T}$ denoted by the form $\mathbb{B}$). In this case the mentioned part denotes that form $\mathbb{D}$ of the theory $\mathscr{T}_{a\mathrm{S}}$ which is obtained by eliminating from this part the direct substitution quantifiers. Later, in the process of reconstructing a form of the theory $\mathscr{T}$ from the form $\mathbb{B}$, the first step of the transformation of the form (part) $\mathbb{D}$ will be the carrying out of the appropriate direct substitution.

Thus a form of the theory $\mathscr{T}_{a\mathrm{R}}$ may, at will, be regarded both as a form of the theory $\mathscr{T}_a$ and as a form of the theory $\mathscr{T}$. Likewise, a form of the theory $\mathscr{T}_{a\mathrm{SR}}$ may be regarded both as a form of the theory $\mathscr{T}_{a\mathrm{S}}$ and as a form of the theory $\mathscr{T}$.

The axioms and rules of inference in the theories $\mathscr{T}_{a\mathrm{S}}$, $\mathscr{T}_{a\mathrm{R}}$, $\mathscr{T}_{a\mathrm{SR}}$ (in particular $\widetilde{\mathscr{T}}_{\mathrm{S}}$, $\widetilde{\mathscr{T}}_{\mathrm{R}}$, $\widetilde{\mathscr{T}}_{\mathrm{SR}}$, $\mathscr{T}_{\mathrm{S}}$, $\mathscr{T}_{\mathrm{R}}$, $\mathscr{T}_{\mathrm{SR}}$) are naturally induced on the basis of the agreement adopted in 2.5.1.

Remark. The notation $\widetilde{\mathscr{T}}_{\mathrm{S}}$ is to be understood as $\mathscr{T}_{a\mathrm{S}}$, where $\mathscr{T}_a$ is $\widetilde{\mathscr{T}}$. The notations $\widetilde{\mathscr{T}}_{\mathrm{R}}$ and $\widetilde{\mathscr{T}}_{\mathrm{SR}}$ are understood similarly. In the case of $\widetilde{\mathscr{T}}_{\mathrm{R}}$ the symbol $\widetilde{\mathscr{T}}$ may be understood also as the basic theory. This does not change the meaning of the notation $\widetilde{\mathscr{T}}_{\mathrm{R}}$.

3.2.2

We have become acquainted with the substitution operator signs in a certain sense in extreme cases: by the first, substitutions are carried out in forms of the (basic) theory $\mathscr{T}$, and by the second, substitutions are carried out in forms of the last (maximal)

extension of the theory $\mathscr{T}$.

In a more general case, as substitution operator signs we shall often use symbols of the form $\mathsf{S}^{n+1}_{\mathscr{T}_c}$, where $\mathscr{T}_c$ is some extension of the theory $\mathscr{T}$ of the kind considered above. The quantifiers constructed using this operator sign indicate that one should carry out substitutions of forms of the theory $\mathscr{T}_c$ in forms of the theory $\mathscr{T}_c$, also in the case when these forms are denoted, for example, by forms of some theory which is an extension of the theory $\mathscr{T}_c$. In accordance with this general agreement, instead of the symbol S^{n+1} one may write $\mathsf{S}^{n+1}_{\mathscr{T}}$. To the two cases of the use of the symbol R^{n+1} considered above (in the theory $\mathscr{T}_{a\mathsf{R}}$ and in the theory $\mathscr{T}_{a\mathsf{SR}}$) there correspond the symbols $\mathsf{S}^{n+1}_{\mathscr{T}_a}$ and $\mathsf{S}^{n+1}_{\mathscr{T}_{a\mathsf{S}}}$.

To illustrate the last agreement let us consider the following example of a general character.

Let each member of the sequence of theories $\mathscr{T} = \mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_m$ (beginning with the second) be such a proper extension of the preceding member that its forms are denotations of forms of the indicated preceding member, and let $\mathscr{T}_{m_0}$ be such a quasi-extension of the theory $\mathscr{T}_m$ that the alphabet of $\mathscr{T}_{m_0}$ is obtained from the alphabet of $\mathscr{T}_m$ by adding the operator signs $\mathsf{S}^{n+1}_{\mathscr{T}_i}$ $(n = 1, 2, \ldots;\ i = 1, 2, \ldots, m)$. The case $m = 1$ is not excluded. Here we suppose that none of the symbols $\mathsf{S}^{n+1}_{\mathscr{T}_i}$ $(n = 1, 2, \ldots;\ i = 1, 2, \ldots, m)$ occurs in the alphabet of the theory $\mathscr{T}_m$. Let now $\mathbb{B}$ be a form of the theory $\mathscr{T}_{m_0}$, and let $\mathsf{S}^{n+1}_{\mathscr{T}_i} X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$ be such a part of the form $\mathbb{B}$ that in the scope of the operator $\mathsf{S}^{n+1}_{\mathscr{T}_i} X_1 \cdots X_n$ there occurs no quantifier of the shape $\mathsf{S}^{k+1}_{\mathscr{T}_l} Y_1 \cdots Y_k$ $(k = 1, 2, \ldots;\ l = 1, 2, \ldots, m)$. Then the part $\mathsf{S}^{n+1}_{\mathscr{T}_i} X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$ denotes the result of the substitution of type $(—/—)$, in the form $\mathbb{A}'$, of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ in place of the letters $X_1, \ldots, X_n$, where $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathbb{A}'$ are such forms of the theory $\mathscr{T}_i$ of which $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{A}$ are denotations. If we replace the part of the form $\mathbb{B}$ under consideration by this result of substitution, we obtain a form $\mathbb{B}_1$ of the theory $\mathscr{T}_{m_0}$ in which the number of quantifiers of the shape $\mathsf{S}^{k+1}_{\mathscr{T}_l} Y_1 \cdots Y_k$ is one less than in $\mathbb{B}$. In the same way from $\mathbb{B}_1$ we obtain $\mathbb{B}_2$, and so on. This process ends with the obtaining of such a form $\mathbb{B}_p$ of the theory $\mathscr{T}_m$ of which the form $\mathbb{B}$ is a denotation. The form $\mathbb{B}$ **directly denotes** the form $\mathbb{B}_p$. If $m > 1$, then the form $\mathbb{B}$ **indirectly denotes** that form $\mathbb{B}'$ of the theory $\mathscr{T}$ of which $\mathbb{B}_p$ is a denotation (the case when $\mathbb{B}_p$ is $\mathbb{B}'$ is not excluded).

Let us consider one more example of a general character. Let each member of the sequence of theories $\mathscr{T} = \mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_m$, $m \geq 1$, be such an extension of the preceding member (theory) that its forms are denotations of forms of the mentioned preceding member (the case $\mathscr{T}_1 = \mathscr{T}_2 = \cdots = \mathscr{T}_m$ is not excluded). Suppose that $\mathscr{T}_{m_0}$ is $\mathscr{T}_m$, and that $\mathscr{T}_{m_1}, \ldots, \mathscr{T}_{m_\ell}$ are defined by the following condition. The theory $\mathscr{T}_{m_i}$ $(i = 1, 2, \ldots, \ell)$ is such a quasi-extension of the theory $\mathscr{T}_{m_{i-1}}$ that its alphabet is obtained from the alphabet of the theory $\mathscr{T}_{m_{i-1}}$ by adding the substitution operator signs $\mathsf{S}^{n+1}_{\mathscr{T}_{m_{i-1}}}$, $n = 1, 2, \ldots$ In addition, suppose that none of the symbols

$$\mathsf{S}^{n+1}_{\mathscr{T}_{m_i}} \quad \text{and} \quad \mathsf{S}^{n+1}_{m_i} \qquad (n = 1, 2, \ldots;\ i = 0, 1, \ldots, \ell - 1)$$

occurs in the alphabet of the theory $\mathscr{T}_m$. For simplicity, instead of the symbols $\mathsf{S}^{n+1}_{\mathscr{T}_{m_i}}$,

$i = 0, 1, \dots, \ell - 1$, we shall also use the symbols $\mathsf{S}^{n+1}_{m_i}$, $i = 0, 1, \dots, \ell - 1$. Let now $\mathbb{B}$ be a form of the theory $\mathscr{T}_{m_\ell}$, and let $\mathsf{S}^{n+1}_{m_i} X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$ be such a part of the form $\mathbb{B}$ that the quantifier $\mathsf{S}^{n+1}_{m_i} X_1 \cdots X_n$ does not lie in the scope of an operator of the shape $\mathsf{S}^{k+1}_{m_j} Y_1 \cdots Y_k$ ($j > i$; $k = 1, 2, \dots$) and, moreover, in the scope of the operator $\mathsf{S}^{n+1}_{m_i} X_1 \cdots X_n$ there is no quantifier of the shape $\mathsf{S}^{k+1}_{m_j} Y_1 \cdots Y_k$ ($j \geq i$; $k = 1, 2, \dots$). Then the part $\mathsf{S}^{n+1}_{m_i} X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$ of the form $\mathbb{B}$ denotes the result of the substitution of type (—/—), in the form $\mathbb{A}$, of the forms $\mathbb{A}_1, \dots, \mathbb{A}_n$ in place of the letters $X_1, \dots, X_n$. If we replace the part of the form $\mathbb{B}$ under consideration by this result of substitution, we obtain a form $\mathbb{B}_1$ of the theory $\mathscr{T}_{m_\ell}$. If by the same method from $\mathbb{B}_1$ we obtain $\mathbb{B}_2$, and so on, then the process ends with the obtaining of a form $\mathbb{B}_p$ of the theory $\mathscr{T}_m$. The form $\mathbb{B}$ directly denotes the form $\mathbb{B}_p$ of the theory $\mathscr{T}_m$. If $m > 1$, then the form $\mathbb{B}$ indirectly denotes the form $\mathbb{B}'$ of the theory $\mathscr{T}$ for which $\mathbb{B}_p$ is a denotation. If for every $i \in \{2, 3, \dots, m\}$ and for any form $\mathbb{A}$ of the theory $\mathscr{T}_i$ a form of the theory $\mathscr{T}_{i-1}$ of which $\mathbb{A}$ is a denotation is determined uniquely, then the form $\mathbb{B}'$ of the theory $\mathscr{T}$ whose denotation is the form $\mathbb{B}$ is determined uniquely as well.

Remark. In the case of the first example, those substitutions are carried out first whose corresponding quantifiers have minimal scopes (i.e. in whose scope there are no other quantifiers), in spite of the lower indices of the corresponding substitution signs. The difference between the substitutions consists only in that each of them acts only on forms of the corresponding theory $\mathscr{T}_i$ (every operand (a form of the theory $\mathscr{T}_m$) of the corresponding substitution quantifier which is not a form of the theory $\mathscr{T}_i$ must be replaced by the form of the theory $\mathscr{T}_i$ denoted by this operand). In the case of the second example the order among the substitutions is determined by more complicated rules.

In the case of both examples the parts (operands) $\mathbb{A}_1, \dots, \mathbb{A}_n, \mathbb{A}$ of the part $\mathsf{S}^{n+1} X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$ we call **substituends**, where the parts $\mathbb{A}_1, \dots, \mathbb{A}_n$ we call **preceding substituends** and $\mathbb{A}$ the **closing substituend**. In the case of the first example, the forms $\mathbb{A}'_1, \dots, \mathbb{A}'_n, \mathbb{A}'$, identifying them with the parts $\mathbb{A}_1, \dots, \mathbb{A}_n, \mathbb{A}$, we call **actual substituends**; $\mathbb{A}'$ is the **actual closing substituend** and the others are the **actual preceding substituends**. In the case of the second example the forms $\mathbb{A}_1, \dots, \mathbb{A}_n, \mathbb{A}$ are also called **actual substituends**. Obviously, in accordance with the agreements adopted above, the **scope of the quantifier** $\mathsf{S}^{n+1}_{\mathscr{T}_i} X_1 \cdots X_n$ consists of the corresponding closing substituend.

Remark. Let $\mathscr{T}_\lambda$ denote either $\mathscr{T}_{m_0}$ from the first example or $\mathscr{T}_{m_\ell}$ from the second example. The reader will easily notice that the structural properties of the forms of the theory $\mathscr{T}_\lambda$ (in particular, of the forms of the theories $\widetilde{\mathscr{T}}_{\mathsf{S}}$, $\widetilde{\mathscr{T}}_{\mathsf{R}}$, $\mathscr{T}_{\mathsf{S}}$, $\mathscr{T}_{\mathsf{R}}$, $\mathscr{T}_{\mathsf{SR}}$) are in the main the same as in the case of the theory $\mathscr{T}$. Likewise, the notions connected with forms of the theory $\mathscr{T}$ are naturally generalized to forms of the theory $\mathscr{T}_\lambda$ in such a way that their essential properties will be preserved. We shall not give definitions of such generalized notions and formulations of their properties, but we shall apply them (and this circumstance will cause no difficulties). This remark relates also to the notions and properties given below.

By means of an example it is easy to show that the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}}$ denoted by the form $\mathrm{R}\,x\,\mathbb{A}_1\,\mathrm{R}\,y\,\mathbb{A}_2\,\mathbb{A}$ of the theory $\widetilde{\mathscr{T}_{\mathrm{R}}}$ may fail to coincide (even up to congruence) with the form $\mathbb{B}'$ of the theory $\widetilde{\mathscr{T}}$ denoted by the form $\mathbb{B}_1$, where $\mathbb{B}_1$ is the result of the substitution, in the form $\mathrm{R}\,y\,\mathbb{A}_2\,\mathbb{A}$, of the form $\mathbb{A}_1$ in place of x (it is assumed that $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}$ are forms of the theory $\mathscr{T}$ containing no substitution operators). In other words: in general, the order of substitutions cannot be changed.

Taking into account the agreement by which $(\mathbb{A}_1, \ldots, \mathbb{A}_n/X_1, \ldots, X_n)\mathbb{A}$ is a simplified notation of the form $\mathrm{S}\,X_1 \cdots X_n \mathbb{A}_1 \cdots \mathbb{A}_n \mathbb{A}$, it is easy to see that the notion of a form of the theory $\mathscr{T}$ constructed using the metavariables $\sigma_1, \ldots, \sigma_n$ and substitutions of type $(—/—)$ relative to object and propositional letters coincides with the notion of a form of the theory $\mathscr{T}_{\mathrm{S}}$ constructed using the metavariables $\sigma_1, \ldots, \sigma_n$. From this follows the validity of the complete analogues of this assertion relating to formulas and terms.

Now the validity of the following theorem is easily noticed.

Theorem 3.1. *Let* $\mathsf{a}_1, \ldots, \mathsf{a}_m$ *be metaletters and let* $\mathbb{A}_1, \ldots, \mathbb{A}_n$ *be such metavariables that for each* $i \in \{1, 2, \ldots, n\}$ *the domain of values of the metavariable* $\mathbb{A}_i$ *is either a set of formulas of the given theory or a set of terms of the given theory. Further, let* $\mathbb{A}$ *be a form constructed using the metavariables*

$$\mathsf{a}_1, \ldots, \mathsf{a}_m,\ \mathbb{A}_1, \ldots, \mathbb{A}_n, \tag{1}$$

and let $\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n$ *and* $\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n$ *be such systems of values of the metavariables* (1) *that*

$$\mathbb{A}'_i \simeq \mathbb{A}''_i, \qquad i = 1, 2, \ldots, n.$$

Then upon substitution in the form $\mathbb{A}$*, in place of the metavariables* (1)*, of the indicated systems of their values, mutually congruent forms* $\mathbb{A}'$ *and* $\mathbb{A}''$ *are obtained.*

Indeed, $\mathbb{A}''$ is obtained from $\mathbb{A}'$ by replacing the substituted occurrences of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ by the forms $\mathbb{A}''_1, \ldots, \mathbb{A}''_n$, respectively. Therefore this theorem is a simple corollary of Theorem 2.5.

Chapter 4

Types of Definitions of Contracting Symbols

In the theory $\mathscr{T}$ (resp. $\mathscr{T}'$, resp. $\mathscr{T}''$) we shall deal with definitions of the form

$$\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}.$$

It is assumed that the general conditions (requirements) given in 2.5 are satisfied. In addition we shall assume that in the right-hand side of each definition only previously introduced contracting symbols may occur. Below, various types of definitions are introduced by means of additional requirements relating to the right-hand side of the definition. For simplicity we shall assume that the only metavariables that may occur in the definitions are $\mathbb{A}_i$, a_j, b_l $(i, j, l = 1, 2, \dots)$ and that these symbols do not occur in the alphabets of the theories under consideration (the metavariable $\mathbb{B}$ does not figure in the definition of a contracting symbol). In practice this restriction is often violated, and arbitrary symbols not occurring in the alphabets of the theories under consideration are used as metavariables in definitions. The results obtained remain valid in the general case, as the reader will easily notice, at least with slight and obvious modifications.

4.1 Definitions of Contracting Symbols of Type I

4.1.1

Type I. The following conditions are required to hold.

Condition (a). The right-hand side of the definition is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables $\mathsf{a}_1, \dots, \mathsf{a}_m, \mathbb{A}_1, \dots, \mathbb{A}_n$ and containing these metavariables, in which all quantifier letters present in explicit form, as well as the metaletters $\mathsf{a}_1, \dots, \mathsf{a}_m$, are bound. Moreover, there is such a subsequence $n_1, \dots, n_\ell$ of the sequence $1, 2, \dots, n$ that each of the metavariables $\mathbb{A}_{n_1}, \dots, \mathbb{A}_{n_\ell}$ lies in the scope of such quantifiers among whose operator letters occur the metaletters $\mathsf{a}_1, \dots, \mathsf{a}_m$ and only they, while the remaining metavariables $\mathbb{A}_i$ do not lie in the scope

of any quantifier. Moreover, the sequence $n_1, \ldots, n_\ell$ is empty if and only if $m = 0$.

Condition (b). If $\mathsf{a}'_1, \ldots, \mathsf{a}'_m$ is any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$, $\mathbb{B}' = (\mathsf{a}'_1, \ldots, \mathsf{a}'_m \,/_{\mathscr{T}:}\, \mathsf{a}_1, \ldots, \mathsf{a}_m)\mathbb{B}$, a is a letter of $\mathbb{B}'$ corresponding to a quantifier letter of the form $\mathbb{B}$ (we mean occurrences of letters), and a^* is a letter of $\mathbb{B}'$ which is a substituted occurrence of some letter a'_i, then neither of the letters a and a^* acts on the other.

Remark. (1) Condition (b) can be expressed in a simplified way as follows: "If a is a letter of the form $\mathbb{B}$ and $i \in \{1, 2, \ldots, m\}$, then neither of the letters a_i and a acts on the other (for any value of the metavariable a_i from its domain of values)".

(2) The same condition (b) can be expressed also as follows: "The groups of bound mathematical letters of the form $\mathbb{B}$ remain groups of bound letters upon passing to the form $\mathbb{B}'$ obtained from $\mathbb{B}$ by substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$" (it is assumed that each metasymbol $\mathbb{A}_i$ is regarded as a propositional letter or an object letter, depending on whether the domain of values of the metavariable $\mathbb{A}_i$ is a class of formulas or a class of terms). If we assume that condition (a) is satisfied, then condition (b) can also be expressed in simplified form as follows: "The groups of bound mathematical letters of the form $\mathbb{B}$ remain groups of bound letters upon substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$, resp. of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to any extended theory $\mathscr{T}_a$".

(3) We shall consider only simultaneous or simple substitutions in place of *all* occurrences of metavariables and, to denote the results of such substitutions, in accordance with the agreements introduced above, we shall use the symbol $(— /_{:} —)$ or the symbols obtained from it by supplying the symbol "/" with indices. In the corresponding terms we shall often omit the clarifying phrases. For example, instead of the phrase "upon substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$ in place of all occurrences of these metavariables" the phrase "upon substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$" will be used.

For the definitions used in practice, condition (b) is usually satisfied trivially: the right-hand sides of the definition contain no quantifier letters, or $m = 0$. An example of the last case is the following definition in the theory $\mathcal{S}$ (see 2.2):

$$\varnothing \text{ ——— } \tau X \forall x\, (x \notin X),$$

where X is the first class variable and x is the first set variable (here X and x are metaconstants and not metavariables; one must always understand in them their values (mathematical letters)). The explicit form of this definition is

$$\varnothing \text{ ——— } \tau_{\mathbb{S}_1} \forall_{\mathbb{S}_7} [\mathbb{S}_7 \notin \mathbb{S}_1].$$

The reconstruction of the explicit form of a definition from such implicit forms requires only overcoming mechanical (inessential) difficulties, and we almost never need such reconstructions.

4.1.2

The class of definitions of type I naturally decomposes into three subclasses, which are determined by the following additional requirements: 1) $m = n = 0$; 2) $m = 0 \wedge n \neq 0$; 3) $m \neq 0$ (and, consequently, $n \neq 0$). The elements of these subclasses we shall call, respectively, **definitions of type** I_1, **of type** I_2 and **of type** I_3.

Obviously, in the case of a definition of type I_1 the left-hand side consists of a single symbol, namely the contracting symbol introduced by this definition, condition (b) drops out, and condition (a) can be formulated as follows.

Condition $(a\,\mathrm{I}_1)$. The right-hand side of the definition is a form of the theory $\widetilde{\mathscr{T}}$ containing no free quantifier letters.

Moreover, it is obvious that in the case of a definition of type I_1 the symbol σ is a derived propositional constant or a derived object constant, depending on whether the form $\mathbb{B}$ is a formula or a term.

In the case of a definition of type I_2 the left-hand side has the form $\sigma\mathbb{A}_1 \cdots \mathbb{A}_n$ (resp. $[\sigma\mathbb{A}_1 \cdots \mathbb{A}_n]$, resp. $[\mathbb{A}_1\,\sigma\,\mathbb{A}_2]$ or $[\sigma\mathbb{A}_1 \cdots \mathbb{A}_n]$, according as $n = 2$ or $n \neq 2$).

Condition (b) drops out in this case also, and condition (a) can be stated as follows.

Condition $(a\,\mathrm{I}_2)$. The right-hand side of the definition is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables $\mathbb{A}_1, \ldots, \mathbb{A}_n$ and containing these metavariables, in which every quantifier letter present in explicit form is bound, and the metavariables $\mathbb{A}_i$ do not lie in the scope of quantifiers.

It is easy to see that in the case of a definition of type I_3 the contracting symbol σ will be an operator sign with weight (n, m) and binding indicator $(n_1, \ldots, n_\ell)$.

Examples of definitions of types I_1 and I_2 occur in the con- sidered definitions of the theory $\mathscr{P}'_1$. An example of a definition of type I_3 is the following definition

$$\exists \mathsf{x}\,\mathbb{A} \text{ ——— } \neg\forall \mathsf{x} \neg \mathbb{A}, \tag{1}$$

where x is any object letter (i.e. x is a metavariable whose set of values is the class of all object letters) and $\mathbb{A}$ is a metavariable whose set of values is the class of all formulas of the theory $\mathscr{T}$.

Remark. If we assume that the right-hand side $\mathbb{B}$ of a definition is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ and containing these metavariables, then the remaining requirements in condition (a) of a definition of type I are a necessary and sufficient condition for both sides of the definition to have one and the same system of free quantifier letters for any choice of a system of values of the metavariables relative to an arbitrary theory $\widetilde{\mathscr{T}_a}$ which is an extension of the theory $\widetilde{\mathscr{T}}$. The last condition is, on the other hand, sufficient for the value of the right-hand side of the definition, when the values of the free variables are fixed, to be determined by the forms $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of the theory $\widetilde{\mathscr{T}_a}$ when the forms $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_\ell}$ are regarded as schemes relative to the letters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ (this does not mean that the indicated schemes will necessarily play the role of arguments of operator signs, but by these schemes those schemes are determined which play the role of argu-

ments of operator signs). It follows that a definition of type I uniquely determines the content of the corresponding contracting symbol (a derived operator or a derived operator sign). But the indicated content will in general depend on the system of values of the non-quantifier constants occurring in the right-hand side of the definition (and will not depend on the system of values of the quantifier constants). This does not exclude the possibility of having such definitions as give the corresponding contracting symbols a content independent of the system of values of the non-quantifier constants occurring in the right-hand side of the definition. An example of such a definition is

$$t \text{ ——— } \mathsf{P} \vee \neg \mathsf{P}, \tag{2}$$

where P is a non-quantifier propositional constant with the set of values $\{t, f\}$.

4.2 Definitions of Contracting Symbols of Types II and II′

Type II. The following conditions are required to hold.

Condition (a_1). The right-hand side of the definition is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m,\ \mathbb{A}_1, \ldots, \mathbb{A}_n,\ \mathsf{b}_1, \ldots, \mathsf{b}_k \tag{1}$$

and containing these metavariables, where b_i is a metaletter whose domain of values is either the class of all quantifier letters of some type or the class of all variables of some type $(i = 1, 2, \ldots, k)$. Moreover, every quantifier letter of the form $\mathbb{B}$ present in explicit form, as well as the metaletters $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathsf{b}_1, \ldots, \mathsf{b}_k$, are bound. Besides, there is such a subsequence $n_1, \ldots, n_\ell$ of the sequence $1, 2, \ldots, n$ that each of the metavariables $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_\ell}$ lies in the scope of such quantifiers among whose operator letters occur the metaletters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ and, besides these metaletters, among the operator letters of the indicated quantifiers only some of the metaletters $\mathsf{b}_1, \ldots, \mathsf{b}_k$ may occur, and each of the remaining metavariables $\mathbb{A}_i$ may lie in the scope only of such quantifiers among whose operator letters occur only some of the metaletters $\mathsf{b}_1, \ldots, \mathsf{b}_k$. Moreover, the sequence $n_1, \ldots, n_\ell$ is empty if and only if $m = 0$. Finally, the order in the sequence $\mathsf{b}_1, \ldots, \mathsf{b}_k$ among the metaletters b_i is naturally determined by their first occurrences in the right-hand side of the definition.

Condition (b). (The same as in the definition of type I.)

Condition (d). The metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$, unlike the other metavariables, do not take their values independently of the values of the other metavariables. Namely, a system

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m,\ \mathbb{A}'_1, \ldots, \mathbb{A}'_n,\ \mathsf{b}'_1, \ldots, \mathsf{b}'_k$$

of values of the sequence of metavariables (1), resp. of values of the sequence of metavariables (1) relative to any extended theory, is admissible if and only if the following two conditions hold.

(d_1). The letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ differ from each other and from the letters $\mathsf{a}'_1, \ldots, \mathsf{a}'_m$, and are not free letters of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$, and none of them has an occurrence (free or bound) in the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables.

(d_2). The indices of the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ are minimal (it is understood that after the choice of the letters $\mathsf{a}'_1, \ldots, \mathsf{a}'_m$ and the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$, the letter b'_1 is first uniquely determined by the conditions (d_1) and (d_2), then, in the same way, by the conditions (d_1) and (d_2) the letter b'_2 is uniquely determined, and so on).

Type II′. To obtain a definition of type II′ it suffices to replace, in the definition of type II, condition (d) by the condition (d') obtained from condition (d) by deleting the condition (d_2).

The metaletters $\mathsf{b}_1, \ldots, \mathsf{b}_k$ figuring in the definitions of contracting symbols introduced above and below we shall call **additional metaletters** (**of the definition**), and their values **additional letters**.

Remark. (1) Condition (b) can be expressed in a simplified way as follows: "If a is a letter of the form $\mathbb{B}$ and $i \in \{1, 2, \ldots, m\}$, then neither of the letters a and a_i acts on the other (for any value of the metavariable a_i from its domain of values)". (2) The same condition (b) can also be expressed as follows. "The groups of bound letters of the form $\mathbb{B}$ remain groups of bound letters upon passing from the form $\mathbb{B}$ to the form $\mathbb{B}'$ obtained from $\mathbb{B}$ by substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$ (it is understood that each metavariable $\mathbb{A}_i$ is regarded as a propositional letter or an object letter, depending on whether the domain of values of the metavariable $\mathbb{A}_i$ is a class of formulas or a class of terms, and each b_i as a letter with the type corresponding to its domain of values)". (3) If we assume that conditions (a_1) and (d') are satisfied (in particular, if we assume that conditions (a_1) and (d) are satisfied), then condition (b) can be expressed in simplified form, namely as follows. "The groups of bound mathematical letters do not change upon substitution of any system of values of the sequence of metavariables (1), resp. of any system of values of the sequence of metavariables (1) relative to any extended theory $\mathscr{T}_a$".

Thus, in contrast to a definition of type II, the choice of a system of values of the metavariables occurring in the left-hand side of a definition of type II′ does not uniquely determine the system of values of the metavariables occurring in the right-hand side of the definition; of the system $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ of values of the metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$ only the satisfaction of condition (d_1) is required.

It is easy to see that for any choice of an admissible system of values of the metavariables relative to any theory that is an extension of the theory $\widetilde{\mathscr{T}}$, the forms obtained in both sides of a definition of any of the considered types I, II and II′ have one and the same system of free quantifier letters. (It is assumed that, analogously to a definition of type I, by a definition of type II in the case $m \neq 0$ a total operator sign or a partial operator sign with binding indicator $(n_1, \ldots, n_\ell)$ is introduced, according as $n_1, \ldots, n_\ell$ is or is not a complete subsequence of the sequence $1, 2, \ldots, n$.)

Remark. In definitions of types II and II′ condition (d_1) can be weakened by replacing it with the following condition (d_0).

Condition (d_0). The substituted occurrences of the letter b'_i in the form

$$\mathbb{B}' = (\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}}}\, \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B},$$

which replace in the form $\mathbb{B}$ the occurrences of the metavariable b_i, form one or several groups of bound letters of the form $\mathbb{B}'$ of the theory $\widetilde{\mathscr{T}}$.

The choice of condition (d_1) instead of condition (d_0) in the case of definitions of type II′ reduces the possibility of writing the given form in contracted form, and in the case of definitions of type II the possibility of the contracted writing of one form is replaced by the possibility of the contracted writing of another form. But this circumstance is not essential. The point is that we almost always consider forms given in contracted form and need to have at our disposal an algorithm for reconstructing a form from its contracted form (and even this, mainly, only in order to *see* the possibility of writing out the (full) forms in place of the contracted forms). Cases in which a form given (written) in full is represented (written) in contracted form are very rare. The choice of condition (d_1) instead of condition (d_0) does not change the class of contracted forms—this class coincides with the class of all forms of the theory $\widetilde{\mathscr{T}}$ (independently of whether we use definitions of type II or definitions of type II′).

The class of definitions of type II also naturally decomposes into three subclasses, which are defined in the same way as the subclasses of the class of definitions of type I. The elements of these subclasses we shall call, respectively, **definitions of type** II_1, **of type** II_2 and **of type** II_3. Obviously each of the types II and II′ is a generalization of type I (a definition of type I corresponds to the case $k = 0$). A definition of type II for which k differs from zero we shall call a **definition of proper type** II. We shall also call it a definition of type II_s. The notion of a **definition of proper type** II_i and the notation $\mathrm{II}_{i\mathrm{s}}$ $(i = 1, 2, 3)$ are defined analogously. Below, introducing some type of definition, we shall assume that thereby subclasses of the class of all definitions of this type have been introduced, defined by the methods considered above. In particular, we shall have analogous subclasses, types and notations connected with the definition of type II′.

Examples of definitions of types $\mathrm{II}_{1\mathrm{s}}$, resp. $\mathrm{II}'_{1\mathrm{s}}$, $\mathrm{II}_{2\mathrm{s}}$, resp. $\mathrm{II}'_{2\mathrm{s}}$, $\mathrm{II}_{3\mathrm{s}}$, resp. $\mathrm{II}'_{3\mathrm{s}}$, written by means of forms of the theory $\widetilde{\mathscr{T}}'$, are:

$$t \text{ ——— } \forall \mathsf{x}(\mathsf{x} = \mathsf{x}),$$

where x is the first, resp. an arbitrary, object variable;

$$[\mathbb{A} \subseteq \mathbb{B}] \text{ ——— } \forall \mathsf{x}\,[\mathsf{x} \in \mathbb{A} \longrightarrow \mathsf{x} \in \mathbb{B}],$$

where $\mathbb{A}$ and $\mathbb{B}$ are arbitrary terms and x is the object variable with minimal index, resp. an arbitrary object variable, having no free occurrences in the terms $\mathbb{A}$ and $\mathbb{B}$;

$$\mathrm{Coll}\,\mathsf{x}\,\mathbb{A} \text{ ——— } \exists \mathsf{y}\,\forall \mathsf{x}\,[\mathsf{x} \in \mathsf{y} \leftrightarrow \mathbb{A}],$$

where Coll is the operator sign introduced by this definition (the group of symbols "Coll" is regarded as a single symbol), x is an arbitrary object variable, $\mathbb{A}$ is an arbitrary formula, and y is the object variable with minimal index, resp. an arbitrary object variable, having no free occurrences in $\mathbb{A}$.

4.3 Definitions of Contracting Symbols of Type III

Let $\widetilde{\mathscr{T}_\mathrm{S}}$ denote the theory whose alphabet is obtained from the alphabet of the theory $\widetilde{\mathscr{T}}$ by adding the substitution operators $\mathrm{S}^{n+1}_{\widetilde{\mathscr{T}}}$, $n = 1, 2, \ldots$. Here, instead of the symbol $\mathrm{S}^{n+1}_{\widetilde{\mathscr{T}}}$ we shall often use the simplified notation $\widetilde{\mathrm{S}}^{n+1}$.

Type III. The conditions (a_S), (b_S) and (c) are required to hold. To obtain the formulations of the conditions (a_S) and (b_S) it suffices to replace the symbols $\widetilde{\mathscr{T}}$ and $/_{\widetilde{\mathscr{T}}}$ by the symbols $\widetilde{\mathscr{T}_\mathrm{S}}$ and $/_{\widetilde{\mathscr{T}_\mathrm{S}}}$ in the formulations of the conditions (a) and (b) of a definition of type I.

Condition (c). Let $\widetilde{\mathscr{T}_a}$ be any extension of the theory $\widetilde{\mathscr{T}}$ whose alphabet contains no substitution operator signs, and let α be any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to the theory $\widetilde{\mathscr{T}_a}$. Then the form of the theory $\widetilde{\mathscr{T}_a}$ obtained by substituting the system α in the left-hand side of the definition, and the form $\mathbb{B}^*$ of the theory $\widetilde{\mathscr{T}_a}$ denoted by the form $\mathbb{B}'_\mathrm{R}$, have one and the same system of free quantifier letters, where $\mathbb{B}'_\mathrm{R}$ is the form of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ obtained from the form $\mathbb{B}$ by replacing every substitution operator sign by the direct substitution operator sign of the same weight and, further, by substituting the system α in the expression $\mathbb{B}_\mathrm{R}$ obtained in this way from $\mathbb{B}$. Let us note here that, according to the agreements introduced above, the symbol $/_{\widetilde{\mathscr{T}_\mathrm{S}}}$ indicates that substitutions of words of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$ are to be carried out in forms of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$, while the substitution operators occurring in forms of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$ indicate that substitutions of forms of the theory $\widetilde{\mathscr{T}}$ are to be carried out in forms of the theory $\widetilde{\mathscr{T}}$.

Remark. In practice condition (c) is almost always equivalent to the following condition.

Condition $(\widetilde{c})$. Let α be any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to $\widetilde{\mathscr{T}}$. Then, upon substitution of the system α in the definition, the form of the theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side and the form of the theory $\widetilde{\mathscr{T}}$ denoted by the form of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$ obtained in the right-hand side have one and the same system of free quantifier letters.

Let us note that $(a_\mathrm{S}) \Rightarrow (c') \wedge (\ell)$, where (c') and (ℓ) are the following conditions.

Condition (c'). Upon substitution of any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to an arbitrary extension $\widetilde{\mathscr{T}_{\mathrm{S}a}}$ of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$, the left-hand side of the definition and the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}_\mathrm{S}}$ give forms of the theory $\widetilde{\mathscr{T}_{\mathrm{S}a}}$ containing one and the same system of free quantifier letters.

Condition (ℓ). Let

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n$$

be admissible systems of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to any extension $\widetilde{\mathscr{T}_a}$ of the theory $\widetilde{\mathscr{T}}$ whose alphabet contains no substitution symbols, with $\mathbb{A}'_i \simeq \mathbb{A}''_i$, $i = 1, 2, \ldots, n$. Further, let $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ be the forms of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ obtained from the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}_{\mathrm{S}}}$ by replacing every substitution operator sign by the direct substitution operator sign of the same weight and then by substituting, in the expression $\mathbb{B}_{\mathrm{R}}$ obtained in this way, the indicated systems of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$. Then $\mathbb{B}'_{\mathrm{R}} \simeq \mathbb{B}''_{\mathrm{R}}$ and, moreover, the forms of the theory $\widetilde{\mathscr{T}_a}$ denoted by the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ are congruent.

Remark. In the case under consideration any system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ is admissible. The word "admissible" is used in condition (ℓ) only for generality of the formulation.

The proof of the metaimplication $(a_{\mathrm{S}}) \Rightarrow (c')$ presents no difficulty. Let us prove $(a_{\mathrm{S}}) \Rightarrow (\ell)$.

Let

$$\mathbb{B}' \rightleftharpoons (\mathsf{a}'_1, \ldots, \mathsf{a}'_m \,/_{\widetilde{\mathscr{T}}_{\mathrm{S}}:}\, \mathsf{a}_1, \ldots, \mathsf{a}_m)\mathbb{B}.$$

Denote by $\overline{\mathbb{B}}'$ the form of the theory $\widetilde{\mathscr{T}_{\mathrm{R}}}$ which is obtained from $\mathbb{B}'$ if every metavariable $\mathbb{A}_i$ $(i = 1, 2, \ldots, n)$ is regarded as a propositional letter or an object letter, according as its domain of values is a class of formulas or a class of terms, and, moreover, if every substitution operator sign of the form $\mathbb{B}'$ is replaced by the direct substitution operator sign of the same weight. Then

$$\mathbb{B}'_{\mathrm{R}} \rightleftharpoons (\mathbb{A}'_1, \ldots, \mathbb{A}'_n \,/_{a\mathrm{R}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n)\overline{\mathbb{B}}';$$
$$\mathbb{B}''_{\mathrm{R}} \rightleftharpoons (\mathbb{A}''_1, \ldots, \mathbb{A}''_n \,/_{a\mathrm{R}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n)\overline{\mathbb{B}}'.$$

Obviously, to obtain the form $\mathbb{B}''_{\mathrm{R}}$ from the form $\mathbb{B}'_{\mathrm{R}}$ it suffices, for each i, to replace each substituted occurrence $\mathbb{A}'_i$ by the form $\mathbb{A}''_i$. Consequently, by virtue of the corollary of Theorem 2.5, we have $\mathbb{B}'_{\mathrm{R}} \simeq \mathbb{B}''_{\mathrm{R}}$.

The mutually corresponding substituted occurrences of the forms $\mathbb{A}'_i$ and $\mathbb{A}''_i$ in the mutually congruent forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ are naturally determined (they replace one and the same occurrence of the letter $\mathbb{A}_i$ in the form $\overline{\mathbb{B}}'$ and are mutually corresponding parts of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$), $i = 1, 2, \ldots, n$. Let $\mathbb{B}'_{\mathrm{R}0}$ and $\mathbb{B}''_{\mathrm{R}0}$ be any mutually corresponding parts of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ that are not proper parts of mutually corresponding substituted occurrences. By the same method by which we proved the congruence of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ it is proved that $\mathbb{B}'_{\mathrm{R}0} \simeq \mathbb{B}''_{\mathrm{R}0}$. Consequently:

(1) mutually corresponding parts of the mutually congruent forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ may fail to be mutually congruent only in the case when they are proper parts of mutually corresponding substituted occurrences.

Now it is also easy to see that

(2) the mutually corresponding substituted occurrences of the mutually congruent forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ contain no direct substitution operators and, consequently, the corresponding parts of the corresponding direct substitution operators of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ are mutually congruent, and the mutually corresponding operands of these direct substitution operators are also mutually congruent;

(3) outside the substituted occurrences the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ coincide.

It suffices to show that the forms of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$ obtained as the result of each pair of corresponding steps of the algorithmic process of reconstructing forms of the theory $\widetilde{\mathscr{T}}_a$ from the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$ are mutually congruent forms of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$.

The mutually corresponding parts of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ on which the substitutions corresponding to the pair of first steps of the said algorithmic processes are carried out have the form

$$\mathbb{J}' \eqsim \mathrm{R}^{k+1}\mathsf{b}_1 \cdots \mathsf{b}_k \mathbb{B}'_1 \cdots \mathbb{B}'_k \mathbb{B}' \quad \text{and} \quad \mathbb{J}'' \eqsim \mathrm{R}^{k+1}\mathsf{b}_1 \cdots \mathsf{b}_k \mathbb{B}''_1 \cdots \mathbb{B}''_k \mathbb{B}''.$$

The forms $\mathbb{J}'_1$ and $\mathbb{J}''_1$ obtained as the result of these substitutions are, by virtue of the first item of Theorem 2.1, mutually congruent. The pair of first steps ends with the replacement of the parts under consideration by the forms $\mathbb{J}'_1$ and $\mathbb{J}''_1$, respectively. Moreover, these parts lie, in a certain sense, in the scope of one and the same letters (the letters of the substituted occurrences of the forms $\mathbb{A}'_i$ and $\mathbb{A}''_i$ do not act on the parts $\mathbb{J}'_1$ and $\mathbb{J}''_1$). Therefore the forms $\mathbb{B}'_{\mathrm{R}1}$ and $\mathbb{B}''_{\mathrm{R}1}$ obtained as the result of the indicated replacements are mutually congruent by virtue of the second item of Theorem 2.2 (the verification of the appropriate conditions presents no difficulty).

To the substituted occurrences of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ lying outside the parts $\mathbb{J}'$ and $\mathbb{J}''$ there naturally correspond parts of the forms $\mathbb{B}'_{\mathrm{R}1}$ and $\mathbb{B}''_{\mathrm{R}1}$. We call them the substituted occurrences of the forms $\mathbb{B}'_{\mathrm{R}1}$ and $\mathbb{B}''_{\mathrm{R}1}$. The notion of mutually corresponding substituted occurrences is also induced naturally. We shall also assume that the parts $\mathbb{J}'_1$ and $\mathbb{J}''_1$ are mutually corresponding substituted occurrences. It is easily verified that the conditions (1)–(3) hold for the forms $\mathbb{B}'_{\mathrm{R}1}$ and $\mathbb{B}''_{\mathrm{R}1}$ as well. This makes it possible to repeat the argument given above in the case of the second pair of steps of the algorithmic process under consideration, and so on.

Remark. $(c') \not\Rightarrow (c)$. Moreover, $(a_{\mathrm{S}}) \wedge (b_{\mathrm{S}}) \not\Rightarrow (c)$. Thus, for example, if in the case of the definition

$$\sigma\, \mathsf{x}\, \mathbb{A}_1 \mathbb{A}_2 \text{ ——— } \forall \mathsf{x}\, \mathsf{S}\, \mathsf{x}\, \mathbb{A}_1 \mathbb{A}_2$$

we take for the values of the metavariable $\mathbb{A}_2$ a form containing no variable, then in the right-hand side we shall have no free variables for any value of $\mathbb{A}_1$ (it is understood that the class of propositional variables is the domain of values of the metavariable x, and the class of all formulas is the domain of values of the metavariables $\mathbb{A}_1$ and $\mathbb{A}_2$).

From what was said in the second remark to the last definition of type I it follows

that any definition of type I is at the same time a definition of type III (i.e. a definition of type I is a particular case of a definition of type III). A definition of type III we call a **definition of proper type** III if it is not a definition of type I. An example of a definition of proper type III is the following definition:

$$\exists \mathsf{X}\, \mathbb{A} \text{ ——— } (\tau \mathsf{X} \mathbb{A}/\mathsf{X})\mathbb{A}, \tag{1}$$

where X is any quantifier object letter, $\mathbb{A}$ is any formula, and $\tau\mathsf{X}$ is a derived or basic logical substantive quantifier. Let us note here that if $\mathsf{x}, \mathsf{y}, \mathsf{z}$ are object letters and $\mathbb{A}$ is $\mathsf{z} = \mathsf{y}$, then (1) gives $\exists \mathsf{x}(\mathsf{z} = \mathsf{y})$ ——— $\mathsf{z} = \mathsf{y}$.

By means of definition (1) the operator sign $\exists$ is often introduced by applying the operator sign τ.

Obviously, the type of definition (1) is none of the types I, II and II′: the right-hand side of this definition is not a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables X and $\mathbb{A}$. Essential here is not the circumstance that the right-hand side of the definition contains a substitution sign at all, but the circumstance that the substitution occurs into a form denoted by a metavariable; essential also is the circumstance that the substitution occurs in place of a letter denoted by a metavariable. Both the first and the second circumstance are sufficient for the right-hand side of the definition not to be a form constructed using the metavariables X and $\mathbb{A}$. The point is that a form constructed using metavariables can always be written in full form (from which the forms written in full form are obtained by substituting systems of values of the metavariables). To write the expression $(\tau\mathsf{X}\mathbb{A}/\mathsf{X})\mathbb{A}$ in such (full) form is, as is easily seen, impossible.

4.4 Definitions of Contracting Symbols of Types IV and IV′

Type IV. The conditions $(a_{1\mathrm{S}})$, (b_{S}), (d_{S}) and (c_1) are required to hold. To obtain the formulations of the conditions $(a_{1\mathrm{S}})$ and (d_{S}) it suffices, in the formulations of the conditions (a_1) and (d) from the definition of type II, to replace the symbols $\widetilde{\mathscr{T}}$ and d_1 by the symbols $\widetilde{\mathscr{T}_{\mathrm{S}}}$ and $d_{1\mathrm{S}}$, and the condition (c_1) is formulated as follows.

Condition (c_1). Let $\widetilde{\mathscr{T}_a}$ be any extension of the theory $\widetilde{\mathscr{T}}$ whose alphabet contains no substitution operator signs, and let α be any admissible system of values of the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m,\ \mathbb{A}_1, \ldots, \mathbb{A}_n,\ \mathsf{b}_1, \ldots, \mathsf{b}_k \tag{1}$$

relative to the theory $\widetilde{\mathscr{T}_a}$. Then the form of the theory $\widetilde{\mathscr{T}_a}$ obtained by substituting the system α in the left-hand side of the definition and the form $\mathbb{B}^*$ of the theory $\widetilde{\mathscr{T}_a}$ denoted by the form $\mathbb{B}'_{\mathrm{R}}$ have one and the same system of free quantifier letters, where $\mathbb{B}'_{\mathrm{R}}$ is the form of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ obtained from the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}_{\mathrm{S}}}$ by replacing every substitution operator sign by the direct substitution operator sign of

the same weight and then by substituting the system α in the expression $\mathbb{B}_{\mathrm{R}}$ obtained in this way from $\mathbb{B}$.

Type IV′. To obtain a definition of this type it suffices, in a definition of type IV, to replace the condition (d_{S}) by the condition (d'_{S}), which is obtained from the condition (d_{S}) by deleting the condition (d_2).

Let us note that in the case of definitions of types IV and IV′ the condition (ℓ_1) holds. More precisely:

$$(a_{1\mathrm{S}}) \wedge (b_{\mathrm{S}}) \wedge [(d_{\mathrm{S}}) \vee (d'_{\mathrm{S}})] \Longrightarrow (\ell_1), \tag{2}$$

where the condition (ℓ_1) is formulated as follows.

Condition (ℓ_1). Let $\widetilde{\mathscr{T}_a}$ be any extension of the theory $\widetilde{\mathscr{T}}$ whose alphabet contains no operator signs of substitution. Further, let

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$$

be such admissible systems of values of the metavariables (1) relative to the theory $\widetilde{\mathscr{T}_a}$ that $\mathbb{A}'_i \simeq \mathbb{A}''_i$ $(i = 1, 2, \ldots, n)$. Further, let $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ be the forms of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ obtained from the form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}_{\mathrm{S}}}$ by replacing every substitution operator sign by the direct substitution operator sign of the same weight and then by substituting, in the expression $\mathbb{B}_{\mathrm{R}}$ obtained in this way, the indicated admissible systems of values of the metavariables (1). Then $\mathbb{B}'_{\mathrm{R}} \simeq \mathbb{B}''_{\mathrm{R}}$ and, moreover, the forms of the theory $\widetilde{\mathscr{T}_a}$ denoted by the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}_{a\mathrm{R}}}$ are congruent.

Remark. Here, in condition (ℓ_1), in the case of a definition of type IV one may write $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ instead of the symbols $\mathsf{b}''_1, \ldots, \mathsf{b}''_k$, since condition (d_{S}) ensures that the letter b''_i coincides with the letter b'_i $(i = 1, 2, \ldots, k)$. In the case of a definition of type IV′ such a replacement yields a true but weaker assertion.

Proof of the metaimplication (2). Let

$$\mathbb{B}' \rightleftharpoons (\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}_{\mathrm{S}}}:} \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B};$$
$$\mathbb{B}'' \rightleftharpoons (\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathsf{b}''_1, \ldots, \mathsf{b}''_k \,/_{\widetilde{\mathscr{T}_{\mathrm{S}}}:} \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B}.$$

Denote by $\overline{\mathbb{B}}'$, resp. $\overline{\mathbb{B}}''$, the form of the theory $\widetilde{\mathscr{T}_{\mathrm{R}}}$ obtained from $\mathbb{B}'$, resp. $\mathbb{B}''$, by replacing every substitution operator sign by the direct substitution operator sign of the same weight and by adopting the agreement by virtue of which every metavariable $\mathbb{A}_i$ is to be regarded as a propositional variable or an object variable, according as the domain of its values is a class of formulas or a class of terms (it would be more precise here to speak, instead of the theory $\widetilde{\mathscr{T}_{\mathrm{R}}}$, of the theory whose alphabet is obtained from the alphabet of $\widetilde{\mathscr{T}_{\mathrm{R}}}$ by adding the metasymbols $\mathbb{A}_1, \ldots, \mathbb{A}_n$ as symbols

of suitable types). Then

$$\mathbb{B}'_{\mathrm{R}} \rightleftharpoons (\mathbb{A}'_1, \ldots, \mathbb{A}'_n \,/_{a\mathrm{R}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n)\overline{\mathbb{B}}',$$
$$\mathbb{B}''_{\mathrm{R}} \rightleftharpoons (\mathbb{A}''_1, \ldots, \mathbb{A}''_n \,/_{a\mathrm{R}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n)\overline{\mathbb{B}}''.$$

Along with these forms let us consider also the form

$$\mathbb{B}^*_{\mathrm{R}} \rightleftharpoons (\mathbb{A}''_1, \ldots, \mathbb{A}''_n \,/_{a\mathrm{R}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n)\overline{\mathbb{B}}'.$$

Between the sets of symbols (occurrences of symbols) of the forms $\mathbb{B}'_{\mathrm{R}}$, $\mathbb{B}''_{\mathrm{R}}$ and $\mathbb{B}^*_{\mathrm{R}}$ a one-to-one correspondence is naturally established (they have one and the same length). The notion of "substituted occurrences" of the letters

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathsf{b}'_1, \ldots, \mathsf{b}'_k, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$$

in the forms $\mathbb{B}'_{\mathrm{R}}$, $\mathbb{B}''_{\mathrm{R}}$ and $\mathbb{B}^*_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$ is likewise naturally defined. Further, from $(a_{1\mathrm{S}}) \wedge (b_{\mathrm{S}}) \wedge (d'_{\mathrm{S}})$ it follows that the substituted occurrences of the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$ in the forms $\mathbb{B}'$, $\mathbb{B}''$, $\mathbb{B}'_{\mathrm{R}}$, $\mathbb{B}''_{\mathrm{R}}$ and $\mathbb{B}^*_{\mathrm{R}}$ make up groups of bound letters (the substituted occurrences of each b'_i, resp. b''_i, make up several groups of bound letters). Moreover, to each such group there is naturally put in correspondence a group of bound letters of the other forms, so that to the group of bound letters of the form $\overline{\mathbb{B}}'$ made up of the substituted occurrences of the letter b'_i there corresponds the group of bound letters of the form $\mathbb{B}'_{\mathrm{R}}$, resp. of the form $\mathbb{B}^*_{\mathrm{R}}$, made up of the substituted occurrences of the letter b'_i. To the same group there corresponds also the group of bound letters of the form $\overline{\mathbb{B}}''$, resp. of the form $\mathbb{B}''_{\mathrm{R}}$, made up of the substituted occurrences of the letter b''_i.

Obviously every occurrence of the letter b'_i or b''_i in the forms $\mathbb{B}^*_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ is either a "substituted occurrence" of this letter or an occurrence in a substituted occurrence of some $\mathbb{A}''_j$. Further, let

$$\mathbb{B}^*_{\mathrm{R}1} \rightleftharpoons \mathbb{B}^*_{\mathrm{R}},\ \mathbb{B}^*_{\mathrm{R}2},\ \ldots,\ \mathbb{B}^*_{\mathrm{R}\ell} \qquad \text{and} \qquad \mathbb{B}''_{\mathrm{R}1} \rightleftharpoons \mathbb{B}''_{\mathrm{R}},\ \mathbb{B}''_{\mathrm{R}2},\ \ldots,\ \mathbb{B}''_{\mathrm{R}\ell}$$

be the sequences corresponding to the parallel algorithmic processes of reconstructing forms of the theory $\widetilde{\mathscr{T}}_a$ from the forms $\mathbb{B}^*_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$. One can define natural numbers $n_1 = k \leq n_2 \leq \cdots \leq n_\ell$, letters $\mathsf{b}'_{n_1+1}, \ldots, \mathsf{b}'_{n_\ell}$, resp. $\mathsf{b}''_{n_1+1}, \ldots, \mathsf{b}''_{n_\ell}$, and their "substituted occurrences" in $\mathbb{B}^*_{\mathrm{R}i}$, resp. in $\mathbb{B}''_{\mathrm{R}i}$ $(i = 1, 2, \ldots, \ell)$ so that the following conditions hold: (1) the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_{n_\ell}$, resp. $\mathsf{b}''_1, \ldots, \mathsf{b}''_{n_\ell}$, are pairwise distinct; (2) every substituted occurrence in $\mathbb{B}^*_{\mathrm{R}j}$, resp. in $\mathbb{B}''_{\mathrm{R}j}$, is an occurrence of some letter from the sequence $\mathsf{b}'_1, \ldots, \mathsf{b}'_{n_j}$, resp. $\mathsf{b}''_1, \ldots, \mathsf{b}''_{n_j}$; (3) to obtain $\mathbb{B}^*_{\mathrm{R}j}$ from $\mathbb{B}''_{\mathrm{R}j}$ it suffices, for each $s \in \{1, 2, \ldots, n_j\}$, to replace every substituted occurrence of the letter b''_s by the letter b'_s, and this causes a simultaneous replacement of groups of bound letters.

Indeed, the "substituted occurrences" of the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$, resp. $\mathsf{b}''_1, \ldots, \mathsf{b}''_k$, in the form $\mathbb{B}^*_{\mathrm{R}1} \rightleftharpoons \mathbb{B}^*_{\mathrm{R}}$, resp. $\mathbb{B}''_{\mathrm{R}1} \rightleftharpoons \mathbb{B}''_{\mathrm{R}}$, are defined above, and the "substituted occurrences" in the form $\mathbb{B}^*_{\mathrm{R}p}$, resp. $\mathbb{B}''_{\mathrm{R}p}$, $1 < p \leq \ell$, are defined by induction as follows. As "substituted occurrences" in the form $\mathbb{B}^*_{\mathrm{R}p}$, resp. $\mathbb{B}''_{\mathrm{R}p}$, we take those and only those

occurrences which naturally correspond to the substitutions needed for the replacement of groups of bound letters and preceding the final immediate substitution in the passage from $\mathbb{B}^*_{\mathrm{R},p-1}$, resp. $\mathbb{B}''_{\mathrm{R},p-1}$, to the form $\mathbb{B}^*_{\mathrm{R}p}$, resp. $\mathbb{B}''_{\mathrm{R}p}$. On the basis of the condition (d'_{S}) it is easy to see that all the "substituted occurrences" defined in this way and all the corresponding letters, suitably denoted, satisfy the conditions (1)–(3).

Then we shall have

$$\mathbb{B}^*_{\mathrm{R}p} \simeq \mathbb{B}''_{\mathrm{R}p} \qquad (p = 1, 2, \ldots, \ell).$$

Consequently, both the forms $\mathbb{B}^*_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$ and the forms of the theory $\widetilde{\mathscr{T}}_a$ denoted by these forms are mutually congruent. Therefore, by virtue of the transitivity of the relation of congruence, it suffices to prove that $\mathbb{B}^*_{\mathrm{R}} \simeq \mathbb{B}'_{\mathrm{R}}$ and that the forms of the theory $\widetilde{\mathscr{T}}_a$ denoted by the forms $\mathbb{B}^*_{\mathrm{R}}$ and $\mathbb{B}'_{\mathrm{R}}$ are mutually congruent. These congruences are justified in the same way as in the above proof of the condition (property) (ℓ) (for definitions of type III), where the congruence of the forms of the theory $\widetilde{\mathscr{T}}_a$ denoted by the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$ of the theory $\widetilde{\mathscr{T}}_{a\mathrm{R}}$, and the congruence of the forms $\mathbb{B}'_{\mathrm{R}}$ and $\mathbb{B}''_{\mathrm{R}}$, were shown.

Remark. Condition (ℓ_1) is satisfied in the case of definitions of types II and II′. Here the condition in the case of definitions of types II and II′ takes the following form.

Condition (ℓ_{10}). Let $\widetilde{\mathscr{T}}_a$ be any extension of the theory $\widetilde{\mathscr{T}}$ whose alphabet contains no operator signs of substitution. Further, let

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$$

be such admissible systems of values of the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k \tag{3}$$

relative to the theory $\widetilde{\mathscr{T}}_a$ that

$$\mathbb{A}'_i \simeq \mathbb{A}''_i, \qquad i = 1, 2, \ldots, n.$$

Then the forms of the theory $\widetilde{\mathscr{T}}_a$ obtained in the right-hand side of the definition upon substitution of the indicated systems of values of the metavariables are mutually congruent.

Likewise, the condition (ℓ) is satisfied in the case of a definition of type I; its simplified formulation (ℓ_0) is obtained from (ℓ_{10}) by deleting the symbols

$$\mathsf{b}'_1, \ldots, \mathsf{b}'_k,\ \mathsf{b}''_1, \ldots, \mathsf{b}''_k,\ \mathsf{b}_1, \ldots, \mathsf{b}_k.$$

It is easy to notice that

$$(a_{1\mathrm{S}}) \Longrightarrow (c'_1) \Longrightarrow (c''_1),$$

where (c'_1) and (c''_1) are the following conditions.

Condition (c_1'). Let α be any admissible system of values of the metavariables (3) relative to the theory $\widetilde{\mathscr{T}}$. Then, upon substitution of the system α in the definition, the form of the theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side and the form $\mathbb{B}'$ of the theory $\widetilde{\mathscr{T}_S}$ obtained in the right-hand side have one and the same system of free quantifier letters.

Condition (c_1''). Let α be any admissible system of values of the metavariables (3) relative to the theory $\widetilde{\mathscr{T}}$. Then the set of free quantifier letters of the form $\mathbb{B}^*$ of the theory $\widetilde{\mathscr{T}}$, denoted by that form $\mathbb{B}'$ of the theory $\widetilde{\mathscr{T}_S}$ which is obtained in the right-hand side of the definition upon substitution of the system α, is a subset of the set of free quantifier letters of the form of the theory $\widetilde{\mathscr{T}}$ obtained by substituting the system α in the left-hand side of the definition.

These conditions are essential for obtaining those results that relate to the case when among the definitions there occur definitions of types II′ and IV′ (i.e. such definitions whose left-hand sides do not uniquely determine the right-hand sides). We have at our disposal a stronger condition, namely the condition (c_1).

4.5 Definitions of Contracting Symbols of Types V, VI, VI′, VII and Relations Between Definitions of Various Types

4.5.1

Type V. The following conditions are required to hold.

Condition (a_2). The right-hand side $\mathbb{B}$ of the definition is a metamathematical expression containing exactly the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n \tag{1}$$

and which, upon substitution of any system of values of these metavariables relative to the theory $\widetilde{\mathscr{T}}$, turns into a denotation of a uniquely determined form of the theory $\widetilde{\mathscr{T}}$ (or turns into a form of the theory $\widetilde{\mathscr{T}}$[1]). Moreover, in the latter form only such contracting symbols and basic or derived operators (including nullary operators) may occur as are in $\mathbb{B}$ or in the components of the system of values of the metavariables (1) under consideration.

Condition (c_2). Let α be any system of values of the metavariables (1) relative to the theory $\widetilde{\mathscr{T}}$. Then, upon substitution of the system α in the definition, the form of the

[1]This form of the theory $\widetilde{\mathscr{T}}$ is to be regarded as a metamathematical expression denoting itself. Further, it is assumed that the rules by which the expression obtained by the substitution under consideration denotes a form of the theory $\widetilde{\mathscr{T}}$ do not depend on the definitions of contracting symbols.

theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side and the form of the theory $\widetilde{\mathscr{T}}$ denoted by the expression $\mathbb{B}'$ obtained in the right-hand side have one and the same system of free quantifier letters.

Condition (ℓ_2). If

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n$$

are such systems of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to the theory $\widetilde{\mathscr{T}}$ that $\mathbb{A}'_i \simeq \mathbb{A}''_i$, $i = 1, 2, \ldots, n$, then the forms of the theory $\widetilde{\mathscr{T}}$ denoted by the expressions obtained upon substitution of the indicated systems in the right-hand side of the definition are mutually congruent.

4.5.2

Type VI. The following conditions are required to hold.

Condition (a_3). The right-hand side $\mathbb{B}$ of the definition is a metamathematical expression containing exactly the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k \tag{1}$$

and which, upon substitution of any system of values of these metavariables relative to the theory $\widetilde{\mathscr{T}}$, turns into a denotation of a uniquely determined form of the theory $\widetilde{\mathscr{T}}$ (or turns into a form of the theory $\widetilde{\mathscr{T}}$[2]). Moreover, in the latter form only such contracting symbols and basic or derived operators (including nullary operators) may occur as are in $\mathbb{B}$ or in the components of the system of values of the metavariables (1) under consideration. In addition, b_i is a metaletter whose domain of values is either the class of all quantifier letters of some type or the class of all variables of some type.

Condition (d^*). The metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$, unlike the other metavariables, do not take their values independently of the values of the other metavariables. Namely, a system

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k$$

of values of the sequence of metavariables (1), resp. of values of the sequence of metavariables (1) relative to the theory $\widetilde{\mathscr{T}}$, is admissible if and only if the following two conditions hold.

(d^*_1). The letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ differ from one another and from the letters $\mathsf{a}'_1, \ldots, \mathsf{a}'_m$, and are not free letters of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$, and none of them occurs in the metamathematical expression $\mathbb{B}$.

(d_2). The indices of the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ are minimal.

Condition (c_3). Let α be any admissible system of values of the metavariables (1)

[2]To weaken the requirement given in this sentence, by requiring its fulfilment only for admissible systems of values of the metavariables (1), has no practical significance.

relative to the theory $\widetilde{\mathscr{T}}$. Then, upon substitution of α in the definition, the form of the theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side and the form of the theory $\widetilde{\mathscr{T}}$ denoted by the expression obtained in the right-hand side have one and the same system of free quantifier letters.

Condition (ℓ_3). If

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \quad \text{and} \quad \mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$$

are such admissible systems of values of the metavariables (1) relative to the theory $\widetilde{\mathscr{T}}$ that $\mathbb{A}'_i \simeq \mathbb{A}''_i$, $i = 1, 2, \ldots, n$, then the forms of the theory $\widetilde{\mathscr{T}}$ denoted by the expressions obtained upon substitution of the indicated systems in the right-hand side of the definition are mutually congruent.

Type VI′. To obtain a definition of this type it suffices, in the definition of type VI, to replace the condition (d^*) by the condition $(d^{*\prime})$, which is obtained from the condition (d^*) by deleting the condition (d_2).

Remark. A complete analogue of the remark given after the formulation of the condition (ℓ_1), relating to definitions of types IV and IV′, holds for the condition (ℓ_3).

4.5.3 Relation between definitions of various types

To contracting symbols and derived operators we relate the type of the corresponding definition. Thus, for example, we may speak of a **derived operator of type** I, a **derived constant of type** II, a **derived operator sign of type** I, etc.

Remark. A number of the results given below we do not prove in the case of contracting symbols of types V, VI and VI′. The reason is that in the definitions of types V, VI and VI′ the characterization of the expression $\mathbb{B}$ is given too generally; these definitions do not specify concretely the ways in accordance with which it denotes a form of the theory $\widetilde{\mathscr{T}}$ (for an arbitrary system of values of the metavariables).

The fact that a definition of type A is a particular case of a definition of type B (i.e. a definition of type A is a definition of type B) we shall write in the form A $\to$ B. We shall use simplified notations connected with this notation. Thus, for example, instead of "A $\to$ B and B $\to$ C" we shall write A $\to$ B $\to$ C.

The relations of the form A $\to$ B between the introduced types of definitions can be presented by the following "graph" (the transitivity of this relation should be borne in mind):

$$\begin{array}{ccccccccc} & & & & \mathrm{II}' & \to & \mathrm{IV}' & \to & \mathrm{VI}' \\ & & & & \uparrow & & \uparrow & & \\ \mathrm{IV} & \leftarrow & \mathrm{II} & \leftarrow & \mathrm{I} & \to & \mathrm{III} & \to & \mathrm{IV} \\ \downarrow & & & & & & \downarrow & & \downarrow \\ \mathrm{VI} & & & & \mathrm{VI}' & \leftarrow & \mathrm{V} & \to & \mathrm{VI} \end{array}$$

Proof. I $\to$ II; I $\to$ II′; III $\to$ IV; III $\to$ IV′; V $\to$ VI; V $\to$ VI′ (it suffices to consider the case $k = 0$). II $\to$ IV; II′ $\to$ IV′ (since the conditions (c_1) are satisfied in the case of definitions of types II and II′). III $\to$ V; IV $\to$ VI; IV′ $\to$ VI′ (follows easily from

the definitions of the corresponding types; we remark that if in conditions (c) and (c_1) we admit that $\widetilde{\mathscr{T}_a}$ is $\widetilde{\mathscr{T}}$, then the substitution operators of the form $\mathbb{B}'$ obtained from $\mathbb{B}$ by substitution of a system of values of the metavariables play the role of direct substitution operators and, consequently, the passage from $\mathbb{B}$ to $\mathbb{B}_{\mathrm{R}}$ becomes superfluous, see the conditions (c), (c_1), (c_2) and (c_3)). Finally, obviously I $\rightarrow$ III.

Let us arrange the various types of definitions in the order of the sequence I, II, II′, III, IV, IV′, V, VI, VI′.

From the graph above it is seen that every type of definition with an odd Roman numeral is a particular case of every subsequent type in the displayed ordering. Moreover, any type of definition with an even Roman numeral and a prime, resp. without a prime, is a particular case of each subsequent type with an even Roman numeral and a prime, resp. without a prime.

A definition of type IV, resp. of type IV′, we call a **proper definition of type** IV, resp. **of type** IV′, if it is neither a definition of type III nor a definition of type II, resp. II′. Likewise, a definition of type VI, resp. of type VI′, we call a **proper definition of type** VI, resp. **of type** VI′, if it is neither a definition of type V nor a definition of type IV, resp. IV′. Obviously a definition of type II, resp. of type II′, is not a definition of type I if and only if it is not a definition of type V (here, instead of the symbol "V", one may write the symbol "III").

An example of a proper definition of type IV, resp. of type IV′, is the following:

$$\exists_1 \mathsf{x}\, \mathbb{A} \text{ ——— } \exists \mathsf{x}\, \mathbb{A} \wedge \forall \mathsf{x}\, \forall \mathsf{y}\, [\mathbb{A} \wedge (\mathsf{y}/\mathsf{x})\mathbb{A} \longrightarrow \mathsf{x} = \mathsf{y}],$$

where x is any object quantifier letter and y is any such object quantifier letter with minimal index (resp. any such object quantifier letter) which differs from x and has no free occurrences in $\mathbb{A}$.

Let us note that the left-hand side of a definition of type II′, resp. IV′, resp. VI′, is uniquely determined by a system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to $\widetilde{\mathscr{T}}$, while the right-hand side depends, moreover, on the system of admissible values of the metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$. It is understood that, by fixing a system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$ relative to the theory $\widetilde{\mathscr{T}}$, the form of the theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side of a definition of type II′, resp. IV′, resp. VI′, denotes any form of the theory $\widetilde{\mathscr{T}}$ denoted by an expression obtained on the right-hand side of the definition by changing the system of values of the metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$ relative to $\widetilde{\mathscr{T}}$ in such a way that, together with the mentioned fixed system of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n$, they give admissible systems of values of the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k$$

relative to $\widetilde{\mathscr{T}}$. In a completely analogous sense, as is easily seen, the condition (d'), resp. (d'_{S}), of a definition of type II′, resp. IV′, ensures that the left-hand side of the definition, after substitution of a system of values of the metavariables contained in it relative to the theory $\widetilde{\mathscr{T}}$, determines the right-hand side up to congruence (the con-

dition (d'), resp. (d'_S), can be weakened, but in such a way that the last requirement is satisfied)—here in the right-hand side of a definition of type II′ forms of the theory $\widetilde{\mathscr{T}}$ are obtained, and in the right-hand side of a definition of type IV′ forms of the theory $\widetilde{\mathscr{T}_S}$. The condition (ℓ_3) of a definition of type VI′ ensures that the left-hand side of the definition, after substitution of a system of values of the metavariables contained in it relative to the theory $\widetilde{\mathscr{T}}$, determines the form of the theory $\widetilde{\mathscr{T}}$ denoted by the right-hand side, up to congruence. Such a conclusion follows directly from the condition (ℓ_1) in the case of a definition of type IV′ (the fulfilment of the condition (ℓ_1) in the case of a definition of type IV′ was proved above). The validity of such a conclusion in the case of a definition of type II′ is obvious (it coincides with the proposition given above concerning a definition of type II′).

In contrast to definitions of types II′, IV′ and VI′, in the case of other definitions, after substitution of a system of values of the metavariables, the form of the theory $\widetilde{\mathscr{T}}$ obtained in the left-hand side uniquely determines the expression obtained in the right-hand side and the form of the theory $\widetilde{\mathscr{T}}$ denoted by this expression.

To obtain the results given below there is no need to retain all the requirements that we imposed on the right-hand sides of definitions. Thus, for example, if one considers a system of definitions each of whose types is I or II or II′, then it suffices to require of the right-hand side of each definition that it be a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables mentioned in the definition and containing these metavariables (which may contain only previously introduced contracting symbols). In particular, this general requirement is satisfied by the so-called definitions of type VII introduced in the next paragraph. But the extension of a number of results to definitions of type VII does not succeed.

In such theories whose alphabets contain no proper propositional constants but contain propositional variables, proper propositional constants are sometimes introduced by means of the following definitions:

$$t \text{ ——— } \mathsf{p} \vee \neg \mathsf{p}; \qquad f \text{ ——— } \mathsf{p} \wedge \neg \mathsf{p},$$

where p is the first propositional variable. Obviously, these definitions belong to none of the types considered above. We shall call them **definitions of type** VII. Most often definitions of type VII are not applied and, consequently, if the alphabet of a given formulation of the propositional calculus contains no proper propositional constants, then derived propositional constants are not introduced (in this case it is impossible to introduce propositional constants by a definition of type I).[3]

The type of every derived operator and contracting symbol is, by definition, induced from the type of the corresponding definition.

[3]The introduction by definitions of type I of the proper propositional constants t and f (in richer theories) by means of compound operators is possible, for example, as follows: t ——— $\forall \mathsf{x}\,[\mathsf{x} = \mathsf{x}]$ and f ——— $\exists \mathsf{x}\,[\mathsf{x} \neq \mathsf{x}]$.

Chapter 5

Definition-Schemes and Contracted Forms

5.1 Instances of Definitions of Contracting Symbols and Derived Operators

5.1.1 Definition-schemes

Often, by means of a definition-scheme, one introduces several or infinitely many contracting symbols depending on some system of indices. Definition-schemes, like definitions, can be of various types. When the system of indices is fixed, a definition-scheme of some type turns into a definition of the same type.

In what follows we shall restrict definition-schemes by certain additional requirements.

Sometimes, for simplicity, the term "definition" will be used instead of the term "definition-scheme" if there is no danger of misunderstanding.

Definition-schemes (among which, in a particular case, some may be definitions) will be numbered by means of words of the form Dm, where m is a natural number. One and the same definition-scheme is usually applied in various theories. A change of the theory causes a change of the meaning of the definition-scheme (without changing the form of the definition-scheme). Therefore, for precision, we shall sometimes write D with indices instead of D in Dm. For example, one and the same definition-scheme will be denoted by Dm or D$'m$ or D$''m$, according to which of the theories $\mathscr{T}$, $\mathscr{T}'$, $\mathscr{T}''$ is meant.

For simplicity we shall assume that for every ℓ, in the right-hand side of every definition taken from the definition-scheme Dℓ there occur only contracting symbols introduced by the preceding definition-schemes, and every contracting symbol is introduced by a unique definition-scheme. We shall also assume that among the

definitions there occurs no so-called **trivial definition** of the form

$$\sigma\,\mathsf{a}_1\cdots\mathsf{a}_m\mathbb{A}_1\cdots\mathbb{A}_n \text{ ——— } \sigma_1\,\mathsf{a}_1\cdots\mathsf{a}_m\mathbb{A}_1\cdots\mathbb{A}_n,$$

where σ_1 is a (basic or derived) operator or operator sign, according as $m = 0$ or $m \neq 0$.

The methods and rules used for representing forms in contracted or simplified form will be applied without change for representing metamathematical expressions in contracted or simplified form.

A definition we shall regard as a particular case of a definition-scheme (as a definition-scheme giving only one definition). We shall assume that in the theory $\mathscr{T}$ the contracting symbols are introduced by means of the following finite or countable sequence of definition-schemes

$$\mathrm{D}1,\ \mathrm{D}2,\ \ldots \tag{1}$$

(the main results obtained below remain valid if, instead of the simple sequence (1), we take any transfinite sequence). It is understood that, to obtain the sequence $\mathrm{D}'1, \mathrm{D}'2, \ldots$, resp. $\mathrm{D}''1, \mathrm{D}''2, \ldots$, of definition-schemes of the theory $\mathscr{T}'$, resp. $\mathscr{T}''$, it suffices to replace in the sequence (1) every form of the theory $\widetilde{\mathscr{T}}$ by the corresponding form of the theory $\widetilde{\mathscr{T}'}$, resp. $\widetilde{\mathscr{T}''}$. Moreover, on the basis of the general agreements adopted in §2.2, (1) can be regarded as a sequence of definition-schemes of the theory $\mathscr{T}'$, resp. $\mathscr{T}''$.

5.1.2 Instances of definitions

The **number of a derived operator**, resp. **of a contracting symbol**, we call the number of the corresponding definition-scheme (by the **definition-scheme corresponding to a derived compound operator** we shall mean the definition-scheme by means of which the operator sign of this compound operator is introduced).

The term "definition of a contracting symbol" (in particular the term "definition of a derived simple operator") needs no further clarification. The expression

$$\sigma\,\mathsf{a}'_1\cdots\mathsf{a}'_m\mathbb{A}_1\cdots\mathbb{A}_n \text{ ——— } \mathbb{B}_1, \tag{1}$$

obtained from the definition

$$\sigma\,\mathsf{a}_1\cdots\mathsf{a}_m\mathbb{A}_1\cdots\mathbb{A}_n \text{ ——— } \mathbb{B} \tag{2}$$

by substituting the system $\mathsf{a}'_1, \ldots, \mathsf{a}'_m$ of values of the metavariables $\mathsf{a}_1, \ldots, \mathsf{a}_m$, we call the **definition of the compound operator** $\sigma\,\mathsf{a}'_1\cdots\mathsf{a}'_m$. The type of this definition is naturally induced from the type of the definition of the contracting symbol σ. The additional metaletters of the definition of the contracting symbol σ we also call the **additional metaletters of the compound operator** $\sigma\,\mathsf{a}'_1\cdots\mathsf{a}'_m$. Further, let $\mathscr{T}_a$ be any extension of the theory $\mathscr{T}$. Any system of values $\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k$ of

the metavariables of the definition (1) of the compound operator $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ we shall call **admissible**, resp. **admissible relative to the theory** $\mathscr{T}_a$, if and only if the system of values

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k$$

of the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k \tag{3}$$

of the definition (2) is admissible, resp. admissible relative to the theory $\mathscr{T}_a$.

It is easy to see that the right-hand side of the definition of a quantifier has the same general properties as the definitions of simple derived operators of the same type; moreover, the admissible systems of values, resp. the systems of values admissible relative to the theory $\mathscr{T}_a$, of the definition of a quantifier can be defined in the same way as in the case of the definition of a simple derived operator of the same type.

Let $\mathbb{A}$ be a form of the theory $\widetilde{\mathscr{T}}$. We shall say that a derived operator σ occurring in $\mathbb{A}$ has a **minimal**, resp. **maximal**, **scope**, if no derived operator occurs in its scope, resp. if it does not occur in the scope of a derived operator.

Let α be a definition of some type, by means of which a contracting symbol, resp. a derived operator, σ is introduced. Such a definition, as we know, has the form

$$\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}, \tag{4}$$

resp.

$$\sigma\, \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}_1, \tag{5}$$

where $\mathbb{B}$, resp. $\mathbb{B}_1$, is a metamathematical expression containing the metavariables

$$\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k, \tag{6}$$

resp.

$$\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k, \tag{7}$$

and only them, in which only previously introduced contracting symbols occur, and which, upon substitution of any system of values of the metavariables relative to the theory $\widetilde{\mathscr{T}}$, turns into a denotation of a uniquely determined form of the theory $\widetilde{\mathscr{T}}$ (or into a form of the theory $\widetilde{\mathscr{T}}$). Moreover, in the latter form there occur only those contracting symbols which are in $\mathbb{B}$, resp. $\mathbb{B}_1$, or in the components of the indicated system of values of the metavariables (other additional conditions are also fulfilled, in particular in special cases—the case is not excluded when the values of all or some of the symbols m, n, k are zero, subject to the condition $m \neq 0 \Rightarrow n \neq 0$). Obviously (5) is a definition of a contracting symbol if and only if σ is not a quantifier.

We call the **canonical quasi-instance of the definition** α the expression obtained from the definition α by substituting into it some admissible system of values of the metavariables (if, for example, $\mathbb{B}$ is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables (6), then the substitution takes place in place of both free and bound

occurrences of the metavariables—the boundness of an occurrence a_0 of a metavariable a means that the substituted occurrence obtained when, in place of the indicated occurrence a_0, the value of the metavariable a is substituted, turns out to be bound. It is assumed that in place of the metavariables there are substituted copies of their values, i.e. copies of words of the theory $\mathscr{T}$). The content of the term **quasi-instance of the definition** α is defined analogously, with the difference that instead of an admissible system of values of the metavariables we take an admissible system of values of the metavariables relative to the theory $\widetilde{\mathscr{T}}$ (i.e. as values of those metavariables whose domains of values are classes of terms of the theory $\mathscr{T}$ or classes of formulas of the theory $\mathscr{T}$, there are allowed forms of the theory $\widetilde{\mathscr{T}}$ of the corresponding types).

Now the meaning of the following terms is clear: **canonical quasi-instance of the definition of a contracting symbol**, resp. **of a derived operator**, and **quasi-instance of the definition of a contracting symbol**, resp. **of a derived operator**.

Obviously the left-hand side of any quasi-instance of a definition α and the form of the theory $\widetilde{\mathscr{T}}$ denoted by the right-hand side of this quasi-instance are forms of the theory $\widetilde{\mathscr{T}}$ of one and the same type.

Obviously every quasi-instance, resp. canonical quasi-instance, of any definition α is a quasi-instance, resp. canonical quasi-instance, of the definition of a uniquely determined derived operator.

If from the introduced terms we delete the particle "quasi", we obtain terms which, by definition, denote the same objects, with the difference that, instead of the expressions occurring in the right-hand sides of the mentioned objects, one should take the forms of the theory $\widetilde{\mathscr{T}}$ denoted by these expressions. Obviously the content of the terms simplified in this way, in the case of definitions of contracting symbols and derived operators of types I, II, II′ and VII, coincides with the content of the corresponding unsimplified terms. This determines the meaning of the following terms: **instance**, resp. **canonical instance**, **of the definition of a contracting symbol** and **instance**, resp. **canonical instance**, **of the definition of a derived operator**. We already know that the main meaning of a definition is expressed by those and only those instances which are at the same time canonical instances.

Let σ be a derived operator sign with the definition (4); let

$$\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k$$

be an admissible system of values of the metavariables (6) of this definition relative to the theory $\widetilde{\mathscr{T}}$; and let

$$\mathbb{B}_1 = (\mathsf{a}'_1, \ldots, \mathsf{a}'_m \,/_{\widetilde{\mathscr{T}}:}\, \mathsf{a}_1, \ldots, \mathsf{a}_m)\mathbb{B};$$
$$\mathbb{B}'_1 = (\mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}}:}\, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B}_1.$$

Then, obviously,

$$\mathbb{B}'_1 = (\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}}:}\, \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B},$$

$$\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}_1 \tag{8}$$

is a definition of the quantifier $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$, and

$$\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m \mathbb{A}'_1 \cdots \mathbb{A}'_n \text{ ——— } \mathbb{B}'_1 \tag{9}$$

is an instance both of the definition (4) and of the definition (8), i.e. (9) is an instance both of the definition of the operator sign σ and of the definition of the quantifier $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$.

In the next section we shall define the meaning of the term **contracted version of a form of the theory** $\mathscr{T}$, and we shall see that the class of contracted versions of forms of the theory $\mathscr{T}$ coincides with the class of all forms of the theory $\widetilde{\mathscr{T}}$, and that every form of the theory $\widetilde{\mathscr{T}}$ is a contracted version of a (in the cases important for practice, uniquely determined, or at least uniquely determined up to congruence) form of the theory $\mathscr{T}$. The contracted versions of forms of the theory $\mathscr{T}$, as said above, are called **contracted forms**.

Now let us introduce several notions.

Let σ be a contracting symbol and σ' a contracting symbol, or a basic operator sign, or a basic simple operator, or a basic non-quantifier constant. We shall say that the contracting symbol σ **directly depends on the symbol** σ' if σ' occurs in the right-hand side of the definition of the contracting symbol σ. Further, we shall say that the contracting symbol σ **depends on the symbol** σ' if $\sigma \neq \sigma'$ and there is a sequence of symbols

$$\sigma_1 = \sigma,\ \sigma_2,\ \ldots,\ \sigma_n = \sigma',$$

every member of which, other than the last, is a contracting symbol and directly depends on the next member. The number of symbols on which a given contracting symbol depends is finite (this is proved by mathematical induction on the number of the contracting symbol). If a contracting symbol σ depends, resp. directly depends, on the symbol σ', one also says that σ is a contracting symbol **following**, resp. **directly following**, the symbol σ', and σ' is a **preceding**, resp. **directly preceding**, symbol of the contracting symbol σ.

Now let $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma'\, \mathsf{C}_1 \cdots \mathsf{C}_\ell$ be operators (it is understood that σ and σ' are symbols introduced in the preceding paragraph and, consequently, $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ is a derived operator, while $\sigma'\, \mathsf{C}_1 \cdots \mathsf{C}_\ell$ is either a derived operator, or a basic operator, or a basic non-quantifier constant). We shall say that the derived operator $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ **directly depends on the operator** $\sigma'\, \mathsf{C}_1 \cdots \mathsf{C}_\ell$ if there exists such a (canonical) quasi-instance

$$\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m \mathbb{A}'_1 \cdots \mathbb{A}'_n \text{ ——— } \mathbb{B}'$$

of the definition of the derived operator $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ in whose right-hand side, outside the substituted occurrences of the forms $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$, the operator $\sigma'\, \mathsf{C}_1 \cdots \mathsf{C}_\ell$ occurs. Further, we shall say that a derived operator δ **depends on the operator or on the basic non-quantifier constant** δ' if $\delta \neq \delta'$ and there exists such a sequence of operators

$$\delta_1 = \delta,\ \delta_2,\ \ldots,\ \delta_k = \delta',$$

every member of which, other than the last, is a derived operator and directly depends on the next member. If a derived operator δ depends, resp. directly depends, on the operator δ', one also says that the derived operator δ is a **following**, resp. **directly following**, operator of δ', and δ' is a **preceding**, resp. **directly preceding**, operator of the derived operator δ. The number of operators on which a given derived operator depends may be infinite. It is finite if the type of every definition-scheme is one of the types I, III, V, VII.

Remark. If the operators δ and δ' are also symbols, then the definitions given in the two preceding paragraphs do not contradict each other: the contracting symbol δ depends, resp. directly depends, on the symbol δ' if and only if the derived operator δ depends, resp. directly depends, on the operator δ'.

Let σ be a derived operator sign; δ an operator sign; $\mathsf{a}_1, \ldots, \mathsf{a}_m, \mathsf{b}_1, \ldots, \mathsf{b}_\ell$ quantifier letters; and $\sigma\,\mathsf{a}_1 \cdots \mathsf{a}_m$ and $\delta\,\mathsf{b}_1 \cdots \mathsf{b}_\ell$ quantifiers. We shall say that the **derived quantifier** $\sigma\,\mathsf{a}_1 \cdots \mathsf{a}_m$ **depends**, resp. **directly depends**, **on the operator sign** δ if $\sigma\,\mathsf{a}_1 \cdots \mathsf{a}_m$ depends, resp. directly depends, on at least one quantifier with the operator sign δ. Further, we shall say that the **derived operator sign** σ **depends**, resp. **directly depends**, **on the quantifier** $\delta\,\mathsf{b}_1 \cdots \mathsf{b}_\ell$ if at least one quantifier with the operator sign σ depends, resp. directly depends, on the quantifier $\delta\,\mathsf{b}_1 \cdots \mathsf{b}_\ell$.

In fact we have defined the meaning of the following terms: *a contracting symbol depends, resp. directly depends, on an operator sign, or on an operator, or on a non-quantifier constant; a derived operator depends, resp. directly depends, on an operator sign, or on an operator, or on a non-quantifier constant.*

It is easy to see that the number of simple operators, non-quantifier constants and operator signs on which a given derived operator or derived operator sign depends is finite.

5.2 Applications of Definitions of Contracting Symbols and Contracted Forms

5.2.1 Various types of applications of the definition of a contracting symbol

Let $\mathbb{A}_1$ ——— $\mathbb{A}_2$ be a canonical instance of a definition α taken from the definition-scheme Dℓ, and let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}'$ and $\mathbb{A}'$ be such forms of the theory $\widetilde{\mathcal{T}}$ that $\mathbb{B}$ is obtained from $\mathbb{A}$ by substituting $\mathbb{A}_1$ in place of some occurrence $\mathbb{A}'_2$ of the form $\mathbb{A}_2$ in $\mathbb{A}$; $\mathbb{A}'$ is obtained from $\mathbb{B}$ by substituting $\mathbb{A}_2$ in place of the considered substituted occurrence $\mathbb{A}_1$ in $\mathbb{B}$; $\mathbb{D}$ is obtained from $\mathbb{C}$ by substituting $\mathbb{A}_2$ in place of some occurrence $\mathbb{A}''_1$ of the form $\mathbb{A}_1$ in $\mathbb{C}$; and, finally, $\mathbb{C}'$ is obtained from $\mathbb{D}$ by substituting $\mathbb{A}_1$ in place of the considered substituted occurrence $\mathbb{A}_2$ in $\mathbb{D}$. Further, let σ be the principal operator of the part $\mathbb{A}''_1$ of the form $\mathbb{C}$. Obviously $\mathbb{A}_1$ ——— $\mathbb{A}_2$ is a canonical instance of the definition of the operator σ.

The transformation of the form $\mathbb{A}$ into $\mathbb{B}$ we shall call the **canonical application in the direct direction of the definition** α, or of the definition Dℓ, or of the defi-

nition of the operator σ. Further, the transformation of the form $\mathbb{C}$ into $\mathbb{D}$ we shall call the **canonical application in the inverse direction of the definition** α, or of the definition $\mathrm{D}\ell$, or of the definition of the operator σ. The latter transformation we shall also call the **application to the operator** σ (the occurrence of the operator σ is meant) **of its definition**. Moreover, for precision, in the case of the first, resp. second, transformation we shall speak of the canonical application of the definition to $\mathbb{A}$, resp. to $\mathbb{C}$. The common name of these two transformations will, by definition, be **canonical application** (of the instance $\mathbb{A}_1$ ——— $\mathbb{A}_2$) of the definition α, or of the definition $\mathrm{D}\ell$, or of the definition of the operator σ. If in the sentences just given we delete the word "canonical", we obtain the definition of the content of the following terms: **application in the direct**, resp. **inverse**, **direction of the definition** α, or of the definition $\mathrm{D}\ell$, or of the definition of the operator σ. The transformations considered (also in the last general case) are respectively also called the **application** (of the instance $\mathbb{A}_1$ ——— $\mathbb{A}_2$) of the definition α, or of the definition $\mathrm{D}\ell$, or of the definition of the operator σ (in the direct, resp. inverse, direction).

Analogously the notions of the various kinds of applications of quasi-instances of the definition α, or of the definition $\mathrm{D}\ell$, or of the definition of the operator σ are introduced, provided the type of the definition is one of the types I–IV, II′, IV′ and VII (there is no such possibility in the case of a definition whose type is one of the types V, VI and VI′). Here the application of some quasi-instance of the definition α we shall call a **quasi-application** (of the corresponding type) of the definition α. Thus, for example, we shall have the notion of **quasi-application in the direct direction of the definition** α or, which is the same, of the application in the direct direction of a quasi-instance of the definition α. Quasi-applications of any type must be carried out with a certain caution. They are usually applied to a denotation of a form of the theory $\widetilde{\mathscr{T}}$, and the result is also a denotation of a form of the theory $\widetilde{\mathscr{T}}$ (in a particular case these denotations may also be forms of the theory $\widetilde{\mathscr{T}}$). The following requirement must hold. If, after the application of a quasi-instance of the definition α, the indicated denotations of forms of the theory $\widetilde{\mathscr{T}}$ are imagined as the corresponding forms of the theory $\widetilde{\mathscr{T}}$, then this application of a quasi-instance must turn into a sequence of applications of the corresponding instance, which may be empty.

The "instances of a definition" and "applications of a definition" of various types considered above we shall call, respectively, **instances of the definition with condition** (d_2) and **applications of the definition with condition** (d_2), if the values of the additional metaletters are taken in accordance with the condition (d_2). Analogous notions connected with the condition (d_2) will be used without a special definition. These refined notions, in the case when the definition (of a contracting symbol) has no additional metaletters or when the definition requires the condition (d_2) to be fulfilled, coincide with the corresponding ordinary notions.

It is easy to see that the following theorem holds.

Theorem 5.1. *1. If the type of a definition α is one of the types I–VI, II′, IV′ and VI′, then the right- and left-hand sides of any instance of the definition α are forms of the theory $\widetilde{\mathscr{T}}$ having one and the same system of free quantifier letters. Likewise, if the type of a*

definition α is one of the types I–IV, II′ *and* IV′, *then the right- and left-hand sides of any quasi-instance of the definition α are forms of the theory $\widetilde{\mathscr{T}_S}$ having one and the same system of free quantifier letters.*

2. *The substitutions (replacements) corresponding to applications and quasi-applications of definitions coincide with the corresponding immediate substitutions—the only exception being the applications (or, which is the same, quasi-applications) of operators of type* VII *(for a definition whose type is one of* V, VI *and* VI′ *we have, generally speaking, no quasi-applications).*

Now it is easy to see that if $\mathbb{A}_1$ ——— $\mathbb{A}_2$ is an instance of the definition of such a derived operator whose type is one of I–VI, II′, IV′ and VI′, or is a quasi-instance of the definition of such a derived operator whose type is one of I–IV, II′, IV′, then for the forms $\mathbb{A}$, $\mathbb{A}'$, $\mathbb{C}$ and $\mathbb{C}'$, which are defined analogously to what was said above, we shall have

$$\mathbb{A} \doteq \mathbb{A}' \qquad \text{and} \qquad \mathbb{C} \doteq \mathbb{C}'$$

(these graphic equalities hold also in the case when $\mathbb{A}$ and $\mathbb{C}$ are forms of the theory $\widetilde{\mathscr{T}_S}$). Here we note at once that both $\mathbb{A}$ and $\mathbb{A}'$, and $\mathbb{C}$ and $\mathbb{C}'$, may fail to be graphically equal when $\mathbb{A}_1$ ——— $\mathbb{A}_2$ is an instance of a definition of type VII.

5.2.2 Various types of sequences of forms of the theory $\widetilde{\mathscr{T}}$ and contracted forms

A finite or infinite sequence $\mathbb{A}_1, \mathbb{A}_2, \ldots$ of forms of the theory $\widetilde{\mathscr{T}}$ we call a **regular sequence**, resp. **canonical regular sequence**, if every form $\mathbb{A}_i$, $i > 1$, is obtained from $\mathbb{A}_{i-1}$ (and, consequently, $\mathbb{A}_{i-1}$ is obtained from $\mathbb{A}_i$) by an application, resp. by a canonical application, of some (suitably chosen) definition Dℓ. We shall say that $\mathbb{A}_1, \mathbb{A}_2, \ldots$ is a **canonical direct normal sequence**, resp. **canonical inverse normal sequence**, resp. **direct normal sequence**, resp. **inverse normal sequence**, if every form $\mathbb{A}_i$, $i > 1$, is obtained from $\mathbb{A}_{i-1}$ by a canonical application in the direct direction, resp. by a canonical application in the inverse direction, resp. by an application in the direct direction, resp. by an application in the inverse direction, of some (suitably chosen) definition Dℓ. The common name of a direct normal sequence and an inverse normal sequence will, by definition, be **normal sequence**. The content of the term **canonical normal sequence** is defined in the same way.

Obviously, the inverse sequence of a finite regular [normal] sequence is a regular [normal] sequence. Further, the inverse sequence of a finite direct normal sequence is an inverse normal sequence. Obviously also, if the first [last] member of a normal sequence is a form of the theory $\mathscr{T}$, then it is a direct [inverse] normal sequence. Complete analogues of these propositions hold for canonical regular [normal] sequences.

A form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ we call a **preceding form**, resp. **canonical preceding form**, of a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, if there is an inverse normal sequence, resp.

canonical inverse normal sequence, of the form

$$\mathbb{A}_1, \ldots, \mathbb{A}_n = \mathbb{A}.$$

The form $\mathbb{A}_1$ is one of the preceding forms, resp. one of the canonical preceding forms, of the form $\mathbb{A}_1$, and is called an **improper preceding form** of the form $\mathbb{A}_1$, resp. an **improper canonical preceding form** of the form $\mathbb{A}_1$.

Let us note that a preceding form of a form $\mathbb{A}_1$ may fail to be a canonical preceding form of the form $\mathbb{A}_1$.

After the introduction of the definitions of contracting symbols, taking into account the remark that, instead of a given form, one and the same contracted version is applied independently of whether this form stands by itself or is a part of a more complex (longer) form, as well as the circumstance that the main meaning of a definition is expressed by its canonical instances, it is self-evident that the obtaining of contracted versions of forms and the reconstruction of forms from contracted versions must proceed by canonical applications of definitions. The following definition corresponds to this consideration.

A form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}}$ is called a **contracted version of a form $\mathbb{A}$ of the theory $\mathscr{T}$ relative to the system of definitions** $\mathrm{D}1, \mathrm{D}2, \ldots$, if there exists a finite canonical normal sequence of forms of the theory $\widetilde{\mathscr{T}}$ whose end members are $\mathbb{A}$ and $\mathbb{B}$. Obviously $\mathbb{A}$ is one of the contracted versions of the form $\mathbb{A}$. This contracted version of the form $\mathbb{A}$ is called **improper**. The contracted versions of forms of the theory $\mathscr{T}$ are called **contracted forms** (**contracted formulas** or **contracted terms**).

It follows from the results obtained below that, in the case important for practice, the content of this definition will not change if we delete from it the word "canonical". In a more special, but still practically important case, the content of this definition will not change in this case either if we replace in it the phrase "canonical normal" by the word "regular" (see Theorems 7.8 and 7.9).

Contracting symbols are naturally applied for contracting quasi-forms (the forms of a theory whose alphabet is obtained from the alphabet of the given theory by adding some auxiliary symbols as constants are called **quasi-forms** of the given (basic) theory), and, in accordance with this, the notions "contracted quasi-form", "contracted quasi-term", "contracted version of a quasi-form", etc. are naturally defined.

It follows easily from the definition that the contracted versions of a form $\mathbb{A}$ of the theory $\mathscr{T}$ are forms of one and the same type, and their type coincides with the type of the form $\mathbb{A}$. Below we shall see that, conversely, any form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ is a contracted version of a form of the theory $\mathscr{T}$ (in the cases important for practice, of a form uniquely determined up to congruence, and in a number of important cases, of a uniquely determined form). Here we note that such forms of the theory $\mathscr{T}$ are determined, for example, by means of the process described in the first paragraph of the next section.

Chapter 6

Basic Processes of Reconstructing Forms from a Contracted Form

6.1 β-, β'-, γ- and γ'-Processes of Reconstructing Forms

For a given form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, the finding of such a form $\mathbb{A}$ of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version can be carried out by the following algorithmic process (algorithm). Let us find the derived operator σ of the form $\mathbb{A}_1$ that has a minimal scope and lies to the left of all other such operators, and apply to it its definition with condition (d_2). The resulting form $\mathbb{A}_2$ of the theory $\widetilde{\mathscr{T}}$ we transform in the same way into a form $\mathbb{A}_3$, and so on. This process ends with the obtaining of such a form $\mathbb{A}_n$ of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version ($\mathbb{A}_1, \ldots, \mathbb{A}_n$ is a canonical (inverse) normal sequence). Any such process we call a β**-process**. Further, the process whose description is obtained from the description of the β-process by deleting the phrase "with condition (d_2)" we call a β'**-process**. Obviously every β-process is a β'-process. Such a generalized β'-process may fail to be algorithmic only in the case when among the contracting symbols of the form $\mathbb{A}_1$, or among the contracting symbols on which at least one contracting symbol of the form $\mathbb{A}_1$ depends, there occurs a contracting symbol whose type is one of II$'$, IV$'$ and VI$'$, since in the opposite case both processes coincide. Obviously every β-process and every β'-process gives a canonical inverse normal sequence.

Let us prove the finiteness of the number of steps of any β'-process (in particular, of a β-process).

Denote by ℓ the largest number among all the numbers each of which is zero or the number of some derived operator occurring in $\mathbb{A}_1$. Suppose that k is a natural number different from zero and that the proposition to be proved holds for every such form for which $\ell < k$. It suffices to suppose that $\ell = k$ and to show the finiteness of the number of steps of the process.

As a result of the first step the part $\mathbb{A}_1^1$ of the form $\mathbb{A}_1$ turns into the form (part) $\mathbb{A}_1^2$, and thereby $\mathbb{A}_1$ turns into the form $\mathbb{A}_2$ of the theory $\widetilde{\mathscr{T}}$. The part $\mathbb{A}_1^1$ has the form $\sigma\,\mathbb{B}_1 \cdots \mathbb{B}_n$, where $\mathbb{B}_1, \ldots, \mathbb{B}_n$ are forms of the theory $\mathscr{T}$ (the case $n = 0$ is not

excluded). Therefore the number of every derived operator of the part $\mathbb{A}_1^2$ of the form $\mathbb{A}_2$ is less than the number k. Obviously the parts corresponding to operators with minimal scopes do not pairwise overlap. Therefore, if $\mathbb{A}_1^2$ is not a form of the theory $\mathscr{T}$, then by the next step, by applying to the derived operator occurring in the part $\mathbb{A}_1^2$ its definition, the part $\mathbb{A}_1^2$ turns into a form $\mathbb{A}_1^3$ and thereby the form $\mathbb{A}_2$ turns into a form $\mathbb{A}_3$, and so on. By the inductive hypothesis this process (a part of the whole process) ends with the transformation of the part $\mathbb{A}_1^1$ of the form $\mathbb{A}_1$ into a form $\mathbb{A}_1^s$ of the theory $\mathscr{T}$, and thereby the form $\mathbb{A}_1$ turns into a form $\mathbb{A}_s$. The form $\mathbb{A}_s$ has the same derived operators as $\mathbb{A}_1$, except the operator σ (here we mean not operators but occurrences of operators; there is a natural one-to-one correspondence between the derived operators of the form $\mathbb{A}_s$ and the derived operators of the form $\mathbb{A}_1$ other than σ, and an identification of the corresponding operators). Now the finiteness of the number of steps of the whole process is obvious.

It is easy to see that the number of steps of a β-process and of any β'-process connected with a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ is one and the same.

Let us consider a process more general than the β-process, resp. β'-process. Namely, a γ**-process**, resp. γ'**-process**, which differs from the β-process, resp. β'-process, only in that at each step the derived operator with a minimal scope is chosen arbitrarily. Obviously these generalized processes also give canonical inverse normal sequences. These generalized processes, as can be seen from the theorem given below, are terminating. Consequently, these generalized processes connected with a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ give forms of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version. γ-processes and γ'-processes connected with a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, like β'-processes, generally speaking, determine the sequence $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of forms of the theory $\widetilde{\mathscr{T}}$ non-uniquely, while a β-process determines the corresponding sequence uniquely.

Obviously the sequence of forms of the theory $\widetilde{\mathscr{T}}$ determined by a γ'-process is also a canonical inverse normal sequence. Moreover, in this way any inverse canonical normal sequence whose first member is $\mathbb{A}_1$ and whose last member is a form of the theory $\mathscr{T}$ is obtained (it can be shown that any γ'-process is terminating and, consequently, only sequences of the indicated kinds are obtained in this way). Besides, it is easy to see that if in the indicated process the type of every operator to which its definition is applied is one of I–VII, then the form $\mathbb{A}$ of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version is determined uniquely (one should bear in mind the definition of the meaning of the replacement symbol $(— // —)$, in particular in the case when among the derived operators to which their definitions are applied there occur operators of type VII). This result remains valid also in that case when in the definitions of types I–V some requirements are considerably weakened. Thus, for example, in the definitions of types I and II the phrase "quantifier letter" may be replaced by the word "variable". An analogous remark will hold for some of the theorems given below.

6.2 Basic Properties of β-, β'-, γ- and γ'-Processes

It follows from the subsequent theorem that if, in the γ'-process under consideration, the type of every derived operator to which its definition is applied is one of I–IV, II′, IV′, VII, then the form $\mathbb{A}$ for which $\mathbb{A}_1$ is a contracted version is determined uniquely up to congruence, in the sense that the forms of the theory $\mathscr{T}$ obtained by various γ'-processes are mutually congruent.

Theorem 6.1. *1. Any γ-process connected with a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ is terminating, and the number of its steps is the same as the number of steps of the β-process connected with the form $\mathbb{A}_1$. Moreover, the forms obtained upon termination of different γ-processes connected with the form $\mathbb{A}_1$ are one and the same form (and, consequently, this unique form coincides with the form obtained as a result of termination of the β-process connected with the form $\mathbb{A}_1$).*

2. If the type of every definition-scheme is one of I–IV, II′ and IV′, then any γ'-process connected with a form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ is terminating, and the number of its steps is the same as the number of steps of the β-process connected with the form $\mathbb{A}_1$. Moreover, the forms obtained upon termination of different γ'-processes connected with the form $\mathbb{A}_1$ are mutually congruent forms of the theory $\mathscr{T}$.

Proof. 1. Let $\mathbb{A}_1, \ldots, \mathbb{A}_p$ be the sequence of forms of the theory $\widetilde{\mathscr{T}}$ corresponding to the β-process connected with the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$. Further, let γ^* be any such γ-process connected with the form $\mathbb{A}_1$ that the sequence of its steps other than the first is the sequence of steps of the β-process connected with the form $\mathbb{A}'_2$ obtained as a result of the first step of the process γ^*. First we shall prove that the number of steps of the process γ^* is p and that the last member of the sequence $\mathbb{A}'_1 = \mathbb{A}_1, \mathbb{A}'_2, \ldots, \mathbb{A}'_m$ of forms of the theory $\widetilde{\mathscr{T}}$ corresponding to the process γ^* is $\mathbb{A}_p$.

Let $\sigma_1, \ldots, \sigma_n$ be the complete enumeration of the derived operators of the form $\mathbb{A}_1$ having minimal scopes, ordered by their occurrences in the form $\mathbb{A}_1$ (we mean the enumeration of occurrences of operators), and let $\mathbb{B}_1, \ldots, \mathbb{B}_n$ be the parts of the form $\mathbb{A}_1$ corresponding to these operators. These parts do not pairwise overlap. There is such an $i \in \{1, 2, \ldots, n\}$ that the operator corresponding to the first step of the process γ^* is σ_i. One may assume that $i > 1$ (otherwise the process γ^* is a β-process). As a result of the first step of the process γ^* the part $\mathbb{B}_i$ of the form $\mathbb{A}_1$ turns into a form (part) $\mathbb{B}^*_i$, and thereby the form $\mathbb{A}_1 = \mathbb{A}'_1$ turns into the form $\mathbb{A}'_2$. Let us identify the part $\mathbb{B}_i$ of the form $\mathbb{A}_1$ with the part $\mathbb{B}^*_i$ of the form $\mathbb{A}'_2$. Thereby the form $\mathbb{A}'_2$ is naturally identified with the form $\mathbb{A}_1$. Here "naturalness" means that outside the indicated parts $\mathbb{B}_i$ and $\mathbb{B}^*_i$ the forms $\mathbb{A}_1$ and $\mathbb{A}'_2$ coincide. As a result of this identification those initial steps of the β-process under consideration which precede the step naturally corresponding to the operator σ_i respectively coincide with the initial steps of the process γ^* following the first step. These steps are described below in items (1), (2), ..., $(i-1)$.

Let $\sigma'_1, \ldots, \sigma'_{i-1}$ be the sequence of derived operators of the form $\mathbb{A}_1$ (and, consequently, of the form $\mathbb{A}'_2$) uniquely determined by the following condition. For

every $k \in \{1, 2, \ldots, i-1\}$, σ'_k is the derived operator with a maximal scope in whose scope $\mathbb{B}_k$ lies and in this scope none of the parts $\mathbb{B}_{k+1}, \ldots, \mathbb{B}_n$ lies (in the case of $\mathbb{A}'_2$ the sequence $\mathbb{B}_{k+1}, \ldots, \mathbb{B}_n$ should be understood as the sequence $\mathbb{B}_{k+1}, \ldots, \mathbb{B}_{i-1}, \mathbb{B}^*_i, \mathbb{B}_{i+1}, \ldots, \mathbb{B}_n$); if such a derived operator does not exist, then σ'_k is σ_k. Obviously the operators

$$\sigma'_{k+1}, \ldots, \sigma'_{i-1}, \qquad k \in \{1, 2, \ldots, i-2\},$$

do not lie in the scope of the operator σ'_k.

(1) The steps that identically transform the "common part" $\mathbb{B}'_1$ of the naturally identified forms $\mathbb{A}_1$ and $\mathbb{A}'_2$, with principal operator σ'_1, into one and the same form (part) $\mathbb{B}'_{11}$ of the theory $\mathscr{T}$ and as a result of which the naturally identified forms $\mathbb{A}_{j_1}$ and $\mathbb{A}'_{j_1+1}$ are obtained. Here and below the "natural identifications" of forms are based on the agreement stated above concerning the identification of the parts $\mathbb{B}_i$ and $\mathbb{B}^*_i$ of the forms $\mathbb{A}_1$ and $\mathbb{A}'_2$. The operators of the mutually identified forms $\mathbb{A}_{j_1}$ and $\mathbb{A}'_{j_1+1}$ naturally corresponding to the operators $\sigma'_2, \ldots, \sigma'_{i-1}$ we again denote by the symbols $\sigma'_2, \ldots, \sigma'_{i-1}$. Below we shall use analogous notation without explicitly introducing the corresponding agreements.

(2) If $i > 2$, then the following steps identically transform the "common part" $\mathbb{B}'_2$ of the naturally identified forms $\mathbb{A}_{j_1}$ and $\mathbb{A}'_{j_1+1}$, with principal operator σ'_2, into one and the same form (part) $\mathbb{B}'_{21}$ of the theory $\mathscr{T}$ and as a result of which the naturally identified forms $\mathbb{A}_{j_2}$ and $\mathbb{A}'_{j_2+1}$ are obtained.

Further, we shall have items completely analogous to item (2), so that the number of the last such item will be $(i-1)$ and, as a result of the steps described in this item, the naturally mutually identified forms $\mathbb{A}_{j_{i-1}}$ and $\mathbb{A}'_{j_{i-1}+1}$ of the theory $\widetilde{\mathscr{T}}$ are obtained.

Now it is easy to see that by the next step of the β-process under consideration the part $\mathbb{B}_{i1}$ of the form $\mathbb{A}_{j_{i-1}}$, naturally corresponding to the part $\mathbb{B}_i$ of the form $\mathbb{A}_1$, turns into the form $\mathbb{B}^*_i$, and thereby $\mathbb{A}_{j_{i-1}}$ turns into the form $\mathbb{A}_{j_{i-1}+1}$, i.e.

$$\mathbb{A}_{j_{i-1}+1} = \mathbb{A}'_{j_{i-1}+1}.$$

Consequently, by virtue of the uniqueness of the β-process, we have $m = p$ and $\mathbb{A}'_m = \mathbb{A}_p$.

Let us now consider any γ-process connected with the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$. Denote it by γ_1. From what has been proved it follows that the number of steps of the β-process connected with the form $\mathbb{A}''_2$ obtained as a result of the first step of the process γ_1 is $p-1$, the number of steps of the β-process connected with the form $\mathbb{A}''_3$ obtained as a result of the second step of the process γ_1 is $p-2$, and so on; finally, the number of steps of the β-process connected with the form $\mathbb{A}''_p$ obtained as a result of the $(p-1)$-st step of the process γ_1 is 1 (this is the zero step as a result of which $\mathbb{A}''_p$ is obtained), i.e. $\mathbb{A}''_p$ is a form of the theory $\mathscr{T}$; that is, the number of steps of any γ-process connected with the form $\mathbb{A}_1$ is p. From what has been proved it also follows that the β-processes connected with the forms $\mathbb{A}''_i$ and $\mathbb{A}''_{i+1}$ give one and the

same form of the theory $\mathscr{T}$. Hence, in turn, it follows that the β-process connected with the form $\mathbb{A}_1$ gives the form $\mathbb{A}''_p$. That is, any γ-process connected with the form $\mathbb{A}_1$ gives the same form as the β-process.

2. Denote by γ'_0 some concrete γ'-process, and by γ_0 the corresponding γ-process, defined by induction in the following way.

Let $\mathbb{A}'_1, \ldots, \mathbb{A}'_\ell$ be the sequence of forms of the theory $\widetilde{\mathscr{T}}$ corresponding to the process γ'_0, in the case when such a sequence is finite. Otherwise we shall assume that $\mathbb{A}'_1, \ldots, \mathbb{A}'_\ell$ is a subsequence of the sequence corresponding to the process γ'_0, and $\ell = p$. Further, let k be a natural number, $k < \min(\ell; p)$, and for every natural number $i < k$ let the i-th step of the process γ_0 be uniquely defined and let the form $\mathbb{A}_{i+1}$ of the theory $\widetilde{\mathscr{T}}$ obtained as a result of the mentioned i-th step be congruent to the form $\mathbb{A}'_{i+1}$ (this condition for $k = 1$ is satisfied trivially). Then, by definition, the k-th step of the process γ_0 is the application to the derived operator $\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m$ of its definition, where $\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m$ is that operator of the form $\mathbb{A}_k$ which naturally corresponds to that operator $\sigma'\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ of the form $\mathbb{A}'_k$ to which at the k-th step of the process γ'_0 its definition is applied (it is meant that the occurrences of the operators $\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m$ and $\sigma'\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$, considered as operators, need not differ from each other). The correctness of this inductive definition follows from the fact that (by virtue of Theorem 7.4) the forms of the theory $\widetilde{\mathscr{T}}$ obtained as a result of the indicated k steps of the processes γ'_0 and γ_0 are mutually congruent. Now it is easy to see that $\ell = p$ and that $\mathbb{A}_1, \ldots, \mathbb{A}_p$ is the sequence of forms of the theory $\widetilde{\mathscr{T}}$ corresponding to the process γ_0, and $\mathbb{A}_p \simeq \mathbb{A}'_p$. It remains to apply the transitivity of congruence (by virtue of the first part of the theorem, $\mathbb{A}_p$ is determined uniquely). □

Finally, let us prove the following theorem.

Theorem 6.2. *Any γ'-process is terminating.*

Proof. Let $\mathbb{A}$ be any form of the theory $\widetilde{\mathscr{T}}$ containing a derived operator. Further, let ℓ be the maximum of the numbers of the derived operators occurring in the form $\mathbb{A}$. A natural number n we call a "number corresponding to the form" $\mathbb{A}$ if there exists a non-empty sequence $\mathbb{B}_1, \ldots, \mathbb{B}_m$ of non-overlapping parts of the form $\mathbb{A}$ satisfying the conditions: (1) for every natural number $i \in \{1, 2, \ldots, m\}$, $\mathbb{B}_i$ contains a derived operator and the number of every derived operator of the part $\mathbb{B}_i$ other than the principal operator of the part $\mathbb{B}_i$ is less than ℓ; (2) there is no part of the form $\mathbb{A}$, lying outside the parts $\mathbb{B}_1, \ldots, \mathbb{B}_m$, which contains a derived operator; (3) $n = m + n_1$, where n_1 is the number of derived operators of the form $\mathbb{A}$ lying outside the parts $\mathbb{B}_1, \ldots, \mathbb{B}_m$. Every form that contains a derived operator has at least one corresponding number.

It suffices to prove the validity of the theorem for forms containing derived operators.

The validity of the theorem with respect to a form $\mathbb{A}$ is obvious in the case when the corresponding ℓ is 1 (in this case the number of steps of the γ'-process following the zero step coincides with the number of derived operators occurring in the form $\mathbb{A}$).

Suppose that: (a) $k > 1$ and the theorem holds for all such forms $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ whose corresponding ℓ is less than k.

It suffices to prove that: (a_1) the theorem holds for all such forms $\mathbb{A}$ whose corresponding ℓ is k.

It is easy to see that the proposition (a_1) is valid for all such forms $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ whose corresponding ℓ is k and for which 1 is a corresponding number of the form $\mathbb{A}$.

Suppose that: (b) $p > 1$ and the proposition (a_1) holds for all such forms $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ whose corresponding ℓ is k and each of which has at least one corresponding number less than p.

It suffices to prove that: (b_1) the proposition (a_1) holds for all such forms $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ whose corresponding ℓ is k and for which p is a corresponding number.

Suppose that $\mathbb{A}_1$ is such a form of the theory $\widetilde{\mathscr{T}}$ whose corresponding ℓ is k and for which p is a corresponding number of the form $\mathbb{A}_1$.

It suffices to prove the validity of the theorem for the form $\mathbb{A}_1$.

There exists a sequence $\mathbb{B}_1, \ldots, \mathbb{B}_m$ of parts of the form $\mathbb{A}_1$ having the proper properties. Let γ'_1 be some γ'-process connected with the form $\mathbb{A}_1$. As a result of the initial steps of the process γ'_1 transformations take place in the parts $\mathbb{B}_1, \ldots, \mathbb{B}_m$ until one of these parts (let it be $\mathbb{B}_i$) turns into a form of the theory $\mathscr{T}$ (for the indicated parts the validity of the theorem follows from the assumption (a) and from the fact that derived operators with number ℓ can occur only among their principal operators). As a result of these steps $\mathbb{A}_1$ turns into a form $\mathbb{A}_2$, and the parts $\mathbb{B}_1, \ldots, \mathbb{B}_m$ turn into the parts $\mathbb{B}_{12}, \ldots, \mathbb{B}_{m2}$ of the form $\mathbb{A}_2$. Obviously the part $\mathbb{B}_{i2}$ either lies in the scope of a derived operator of the form $\mathbb{A}_2$, or does not. In the first case, if we denote by $\mathbb{B}'_{i2}$ such a minimal part of the form $\mathbb{A}_2$ containing the part $\mathbb{B}_{i2}$ whose principal operator is a derived operator, then

$$\mathbb{B}_{12}, \ldots, \mathbb{B}_{i-1,2}, \mathbb{B}'_{i2}, \mathbb{B}_{i+1,2}, \ldots, \mathbb{B}_{m2} \tag{1}$$

will be a non-empty sequence of non-overlapping parts of the form $\mathbb{A}_2$ satisfying the following conditions. $(1')$ Any part $\mathbb{B}$ of the form $\mathbb{A}_2$ taken from the sequence (1) contains a derived operator, and the number of every derived operator of this part other than the principal operator is less than ℓ; $(2')$ there is no part of the form $\mathbb{A}_2$, lying outside the parts (1), which contains a derived operator. In the second case the sequence

$$\mathbb{B}_{12}, \ldots, \mathbb{B}_{i-1,2}, \mathbb{B}_{i+1,2}, \ldots, \mathbb{B}_{m2}$$

has the same properties (in this case, since $p > 1$, we have $m > 1$). Consequently, (in both cases) either the ℓ corresponding to the form $\mathbb{A}_2$ is less than k, or ℓ is equal to k and $p - 1$ is a corresponding number of the form $\mathbb{A}_2$. Therefore, by virtue of the assumptions (a) and (b) the theorem is true for the form $\mathbb{A}_2$ and, consequently, for the form $\mathbb{A}_1$. □

Chapter 7

Basic Properties of Contracting Symbols and Contracted Forms. α- and α^*-Processes of Reconstructing Forms from Contracted Forms

7.1 Basic Properties of Contracting Symbols and Contracted Forms

Theorem 7.1. *Let $\mathbb{A}_0$ and $\mathbb{B}_0$ be mutually congruent forms of the theory $\widetilde{\mathscr{T}}$, and let $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ be their mutually corresponding derived operators (the case $m = 0$ is not excluded). Further, let the type of the contracting symbol σ' (and, consequently, also the type of the contracting symbol σ'') be one of* I–IV, II′ *and* IV′, *and let $\mathbb{A}'_0$ and $\mathbb{B}'_0$ be the forms of the theory $\widetilde{\mathscr{T}_\mathsf{S}}$ obtained from the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ by quasi-applications to the operators $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ of their definitions. Then $\mathbb{A}'_0 \simeq \mathbb{B}'_0$.*

Theorem 7.2. *Let $\mathbb{A}$ and $\mathbb{B}$ be mutually congruent forms of the theory $\widetilde{\mathscr{T}_\mathsf{S}}$, and let*

$$\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_m \mathbb{A}_0 \qquad \textit{and} \qquad \widetilde{\mathsf{S}}\,\mathsf{b}_1 \cdots \mathsf{b}_m \mathbb{B}_1 \cdots \mathbb{B}_m \mathbb{B}_0$$

be their mutually corresponding parts whose principal operators are $(m+1)$-place substitution operators with minimal scopes (i.e. in the scope of each of these operators there is no substitution operator). Further, let $\mathbb{A}'$ and $\mathbb{B}'$ be the forms of the theory $\widetilde{\mathscr{T}_\mathsf{S}}$ obtained from the forms $\mathbb{A}$ and $\mathbb{B}$ by replacing the indicated parts by forms of the theory $\widetilde{\mathscr{T}}$ which are the results of the substitutions corresponding to the principal operators of these parts. Then $\mathbb{A}' \simeq \mathbb{B}'$.

Remark. In the preceding theorem the requirement that the scopes of the substitution operators be minimal is not essential. This fact is a simple consequence of the theorem itself, since in the case of such a general formulation the phrase "by replacing the

indicated parts by forms of the theory $\widetilde{\mathscr{T}}$ which are the results of the substitutions corresponding to the principal operators of these parts" requires that into the form of the theory $\widetilde{\mathscr{T}}$ denoted by the form $\mathbb{A}_0$, resp. $\mathbb{B}_0$, there be substituted the forms of the theory $\widetilde{\mathscr{T}}$ denoted by the forms $\mathbb{A}_1, \ldots, \mathbb{A}_m$, resp. $\mathbb{B}_1, \ldots, \mathbb{B}_m$, of the theory $\widetilde{\mathscr{T}_S}$. This is achieved by means of a process consisting of steps, each of which is a substitution by a substitution operator with a minimal scope (it suffices that each pair of steps consist of substitutions by the corresponding substitution operators and to apply the theorem with respect to these substitutions). Here we note that in the mentioned general case the conclusion of the theorem is valid also if, instead of the indicated phrase, we take the phrase: "by replacing the indicated parts by forms of the theory $\widetilde{\mathscr{T}_S}$ which are the results of the substitutions corresponding to the principal operators of these parts relative to the theory $\widetilde{\mathscr{T}_S}$", where the phrase "relative to the theory $\widetilde{\mathscr{T}_S}$" indicates, for example, that one should perform the substitution of the forms $\mathbb{A}_1, \ldots, \mathbb{A}_m$ of the theory $\widetilde{\mathscr{T}_S}$ into the form $\mathbb{A}_0$ of the theory $\widetilde{\mathscr{T}_S}$ (to obtain this consequence from the theorem it suffices to regard the non-principal substitution operators occurring in the parts under consideration as basic operators of the corresponding type, i.e. to regard them as complex quantifiers).

Before the proof of these theorems let us note that the following theorems are their simple consequences.

Theorem 7.3. *The forms $\mathbb{A}$ and $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}}$ denoted by mutually congruent forms $\mathbb{A}_1$ and $\mathbb{B}_1$ of the theory $\widetilde{\mathscr{T}_S}$ are mutually congruent.*

Theorem 7.4. *Let $\mathbb{A}_0$ and $\mathbb{B}_0$ be mutually congruent forms of the theory $\widetilde{\mathscr{T}}$, and let $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ be their mutually corresponding derived operators (the case $m = 0$ is not excluded). Further, let the type of the contracting symbol σ' (and, consequently, also the type of the contracting symbol σ'') be one of* I–IV, II′ *and* IV′, *and let $\mathbb{A}'_0$ and $\mathbb{B}'_0$ be the forms of the theory $\widetilde{\mathscr{T}}$ obtained from the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ by applications to the operators $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ of their definitions. Then $\mathbb{A}'_0 \simeq \mathbb{B}'_0$.*

Theorem 7.5. *Let $\mathbb{A}_0$ and $\mathbb{B}_0$ be mutually congruent forms of the theory $\widetilde{\mathscr{T}}$, and let $\mathbb{A}$ and $\mathbb{B}$ be such forms of the theory $\mathscr{T}$ for which $\mathbb{A}_0$ and $\mathbb{B}_0$ are contracted versions. If the type of every derived operator occurring in $\mathbb{A}_0$ (and, consequently, in $\mathbb{B}_0$), as well as the type of every derived operator preceding some derived operator of $\mathbb{A}_0$, is one of the types*[1] I–IV, II′ *and* IV′, *then $\mathbb{A} \simeq \mathbb{B}$.*

Theorem 7.6. *Let the type of every derived operator occurring in the form $\mathbb{A}_0$ of the theory $\widetilde{\mathscr{T}}$, as well as the type of every derived operator that is a preceding operator of some operator of $\mathbb{A}_0$, be one of* I–IV, VII (*resp. one of* I–IV, II′, IV′). *Then that form $\mathbb{A}$ of the theory $\mathscr{T}$ for which $\mathbb{A}_0$ is a contracted version is determined uniquely (resp. is determined uniquely up to congruence, in the sense that all those forms for which $\mathbb{A}_0$ is a contracted version are mutually congruent).*

[1] Obviously this last condition is fulfilled if the type of every definition-scheme from D1, D2, ... is one of I–IV, II′, IV′.

Remark. The condition, for example, of the first part of this theorem is equivalent to the following condition. The type of every contracting symbol occurring in the form $\mathbb{A}_0$ of the theory $\widetilde{\mathscr{T}}$, as well as the type of every contracting symbol that is a preceding symbol of some contracting symbol of $\mathbb{A}_0$, is one of I–IV, VII. One should bear in mind complete analogues of this remark also in the case of other theorems.

Indeed, to obtain Theorem 7.3 one should apply Theorem 7.2 as many times as a substitution operator occurs in the form $\mathbb{A}_1$ (and, consequently, in the form $\mathbb{B}_1$). To prove Theorem 7.4 it suffices to note that the application of the definition of a derived operator decomposes into a quasi-application of the definition of the same operator and a sequence of eliminations of substitution operators, on the basis of Theorem 7.2, from the substitution part of the form obtained as a result of the indicated quasi-application. Let us give the proof of Theorem 7.5 in more detail.

Let

$$\mathbb{A}_1 = \mathbb{A}_0, \mathbb{A}_2, \ldots, \mathbb{A}_n = \mathbb{A} \qquad \text{and} \qquad \mathbb{B}_1 = \mathbb{B}_0, \mathbb{B}_2, \ldots, \mathbb{B}_m = \mathbb{B}$$

be canonical normal sequences (their existence follows from the definition of a contracted version of a form). Further, let K be such a maximal natural number that for every natural number $i \leq K$ we have $\mathbb{A}_i \simeq \mathbb{B}_i$ and, if $i > 1$, then $\mathbb{A}_i$ and $\mathbb{B}_i$ are obtained from $\mathbb{A}_{i-1}$ and $\mathbb{B}_{i-1}$ by applying their definitions to corresponding operators. Suppose that $K < \min(m; n)$. It suffices to show that the sequence $\mathbb{B}_1, \ldots, \mathbb{B}_m$ can be transformed, without changing the first K members, the last member and the number of members, in such a way that for the new canonical normal sequence the maximal natural number possessing the property under consideration is greater than K (from this it will also easily follow that $m = n$).

Let $\sigma'_1, \ldots, \sigma'_\ell$ and $\sigma''_1, \ldots, \sigma''_\ell$ be such enumerations of all derived operators (of the mutually congruent) forms $\mathbb{A}_K$ and $\mathbb{B}_K$ having minimal scopes that σ'_i and σ''_i are mutually corresponding operators ($i = 1, 2, \ldots, \ell$). Without loss of generality one may assume that $\mathbb{A}_{K+1}$ is obtained from $\mathbb{A}_K$ by applying to the operator σ'_1 its definition, and $\mathbb{B}_{K+1}$ is obtained from $\mathbb{B}_K$ by applying to the operator σ''_2 its definition (see Theorem 7.4). Further, from our assumption it easily follows that $K < \min(m; n) - 1$. Denote by $\mathbb{A}'_i$ and $\mathbb{B}'_i$ the parts of the forms $\mathbb{A}_K$ and $\mathbb{B}_K$ with principal operators σ'_i and σ''_i ($i = 1, 2, \ldots, \ell$). To these parts there naturally correspond transformed or untransformed parts in the subsequent members of the sequences under consideration. Then there will be such a natural number $p > 0$ that the parts of the forms $\mathbb{B}_{K+1}, \ldots, \mathbb{B}_{K+p}$ corresponding to the part $\mathbb{B}'_1$, regarded as forms (of the theory $\widetilde{\mathscr{T}}$), coincide with the form $\mathbb{B}'_1$, while in the passage from the form $\mathbb{B}_{K+p}$ to the form $\mathbb{B}_{K+p+1}$ this corresponding part is transformed; i.e. $\mathbb{B}_{K+p+1}$ is obtained from $\mathbb{B}_{K+p}$ by applying to the operator corresponding to the operator σ''_1 its definition—by means of this application the part corresponding to the part $\mathbb{B}'_1$ is replaced by some form $\mathbb{D}$. In the members $\mathbb{B}_K, \ldots, \mathbb{B}_{K+p}$ we replace the part $\mathbb{B}'_1$ and the parts corresponding to it by the form $\mathbb{D}$ and denote the forms obtained by the symbols $\mathbb{B}^*_{K+1}, \ldots, \mathbb{B}^*_{K+p+1}$

respectively. On the basis of Theorem 7.4 it is easy to see that

$$\mathbb{B}_1, \ldots, \mathbb{B}_K, \mathbb{B}^*_{K+1}, \ldots, \mathbb{B}^*_{K+p+1} \rightleftharpoons \mathbb{B}_{K+p+1}, \mathbb{B}_{K+p+2}, \ldots, \mathbb{B}_m$$

is a canonical normal sequence possessing the required properties (by such transformations one obtains such a canonical normal sequence

$$\overline{\mathbb{B}}_1 \rightleftharpoons \mathbb{B}_1, \overline{\mathbb{B}}_2, \ldots, \overline{\mathbb{B}}_m \rightleftharpoons \mathbb{B}$$

for which the corresponding maximal natural number is $\min(m; n)$ and, consequently, $n = m$).

By analogous reasoning both parts of Theorem 7.6 are proved, if both in the role of $\mathbb{A}_0$ and in the role of $\mathbb{B}_0$ we take $\mathbb{A}_0$, and in the proof of the first part, instead of congruence, we speak of graphic equality (this last circumstance guarantees the inclusion of derived operators of type VII).

Remark. The second part of Theorem 7.6 can be extended also to the case when derived operators of type VII occur in the form $\mathbb{A}_0$, but subject to the condition that among the operators preceding the operators of the form $\mathbb{A}_0$ there occur no operators of type VII. Indeed, in this case, in both sequences (after the transformation of the second sequence), upon the elimination of derived operators of type VII congruence is not violated, since upon the indicated elimination the derived operators (constants) turn out to lie in the scope of "one and the same" letters (derived operators of type VII can be eliminated from the very beginning without changing the totality of the final results). In the case of Theorem 7.5 such a remark is not valid—the indication of a suitable example presents no difficulty.

Proof of Theorem 7.1. Since each definition of types III, II and I is a particular case of a definition of type IV, and a definition of type II′ is a particular case of a definition of type IV′, without loss of generality we may assume that σ is a derived operator or operator sign of type IV or IV′ with the definition

$$\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}_S, \tag{1}$$

where σ is σ' or, which is the same, σ'', regarded as a derived operator or operator sign (according as $m = 0$ or $m \neq 0$).

Suppose that the forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ are obtained from the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ by applying to the operators $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ the following quasi-instances of the definition (1):

$$\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m \mathbb{A}'_1 \cdots \mathbb{A}'_n \text{ ——— } \mathbb{B}'_S, \tag{2}$$

$$\sigma\, \mathsf{a}''_1 \cdots \mathsf{a}''_m \mathbb{A}''_1 \cdots \mathbb{A}''_n \text{ ——— } \mathbb{B}''_S. \tag{3}$$

Obviously, for these quasi-instances the condition (c'_1) of 4.4 is satisfied. Denote by $\mathbb{A}'$ and $\mathbb{B}'$ those parts of the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ whose principal operators are $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$—these, regarded as forms, are the left-hand sides of the quasi-instances (2) and (3) of the definition (1).

Further, let $\mathbb{A}_1, \ldots, \mathbb{A}_p$ be such an enumeration of (some) canonical representatives of groups of bound letters of the form $\mathbb{A}_0$ that is uniquely determined by the condition: any canonical representative $\mathbb{A}$ of a group of bound letters of the form $\mathbb{A}_0$ is a member of the sequence $\mathbb{A}_1, \ldots, \mathbb{A}_p$ if and only if $\mathbb{A}$ acts on the part $\mathbb{A}'$ and, moreover, the order among these representatives in the sequence $\mathbb{A}_1, \ldots, \mathbb{A}_p$ is the same as the order among them in the form $\mathbb{A}_0$. Denote by $\mathbb{B}_1, \ldots, \mathbb{B}_p$ the letters of the form $\mathbb{B}_0$ corresponding to the letters $\mathbb{A}_1, \ldots, \mathbb{A}_p$. Let us note that from $\mathbb{A}_0 \simeq \mathbb{B}_0$ it follows that the parts $\mathbb{A}'$ and $\mathbb{B}'$ are mutually variant (see Theorem 2.6). From the indicated congruence it also follows that the sequence $\mathbb{B}_1, \ldots, \mathbb{B}_p$ can be regarded as an enumeration of canonical representatives of groups of bound letters of the form $\mathbb{B}_0$, defined analogously to the enumeration $\mathbb{A}_1, \ldots, \mathbb{A}_p$. Moreover, mutually corresponding free letters of the parts $\mathbb{A}'$ and $\mathbb{B}'$, other than $\mathbb{A}_1, \ldots, \mathbb{A}_p$ and $\mathbb{B}_1, \ldots, \mathbb{B}_p$ respectively, are one and the same letters, and such and only such letters are free letters of the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ lying in the parts $\mathbb{A}'$ and $\mathbb{B}'$ (we mean occurrences of letters).

By virtue of Theorem 5.1

$$\mathbb{A}'_0 = (\mathbb{B}'_S /\!/_{\widetilde{\mathscr{T}}_S:} \mathbb{A}')\mathbb{A}_0 \qquad \text{and} \qquad \mathbb{B}'_0 = (\mathbb{B}''_S /\!/_{\widetilde{\mathscr{T}}_S:} \mathbb{B}')\mathbb{B}_0.$$

Denote by $\overline{\mathbb{B}}'_S$, resp. $\overline{\mathbb{B}}''_S$, the substituted occurrence of the form $\mathbb{B}'_S$ in $\mathbb{A}'_0$, resp. of $\mathbb{B}''_S$ in $\mathbb{B}'_0$. Further, obviously,

$$\begin{aligned}\mathbb{B}'_S = (&\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathbb{A}'_1, \ldots, \mathbb{A}'_n, \mathsf{b}'_1, \ldots, \mathsf{b}'_k\\ &/_{\widetilde{\mathscr{T}}_S:} \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B}_S,\\ \mathbb{B}''_S = (&\mathsf{a}''_1, \ldots, \mathsf{a}''_m, \mathbb{A}''_1, \ldots, \mathbb{A}''_n, \mathsf{b}''_1, \ldots, \mathsf{b}''_k\\ &/_{\widetilde{\mathscr{T}}_S:} \mathsf{a}_1, \ldots, \mathsf{a}_m, \mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B}_S.\end{aligned}$$

Let us identify with one another the corresponding symbols of the forms $\mathbb{A}_0$ and $\mathbb{A}'_0$, resp. $\mathbb{B}_0$ and $\mathbb{B}'_0$, lying outside the parts $\mathbb{A}'$ and $\overline{\mathbb{B}}'_S$, resp. $\mathbb{B}'$ and $\overline{\mathbb{B}}''_S$—the corresponding symbols will be one and the same symbols.

Let $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$, resp. $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$, be the sequence of occurrences of letters in the form $\mathbb{A}'_0$, resp. $\mathbb{B}'_0$, determined with the help of $\mathbb{A}'_0$ and $\overline{\mathbb{B}}'_S$, resp. $\mathbb{B}'_0$ and $\overline{\mathbb{B}}''_S$, in the same way as $\mathbb{A}_1, \ldots, \mathbb{A}_p$ is determined with the help of $\mathbb{A}_0$ and $\mathbb{A}'$. Now it is easy to see the validity of the following assertions.

(1′) The forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ of the theory $\widetilde{\mathscr{T}_S}$ have one and the same length and (under the canonical one-to-one correspondence between their symbols) the parts $\overline{\mathbb{B}}'_S$ and $\overline{\mathbb{B}}''_S$ correspond to each other in such a way that the substituted occurrences of the parts $\mathbb{A}'_i$ and $\mathbb{A}''_i$ $(i = 1, 2, \ldots, n)$ and of the letters $\mathsf{a}'_1, \ldots, \mathsf{a}'_m, \mathsf{b}'_1, \ldots, \mathsf{b}'_k$ and $\mathsf{a}''_1, \ldots, \mathsf{a}''_m, \mathsf{b}''_1, \ldots, \mathsf{b}''_k$ respectively correspond to each other, while the remaining corresponding occurrences are one and the same symbols. The occurrences $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$ and $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$ likewise correspond to each other respectively.

(2′) On the basis of the agreement adopted above on the identification of certain corresponding occurrences of symbols of the forms $\mathbb{A}_0$ and $\mathbb{A}'_0$, resp. $\mathbb{B}_0$ and $\mathbb{B}'_0$, the oc-

currences $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p, \overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$ and $\mathbb{A}_1, \ldots, \mathbb{A}_p, \mathbb{B}_1, \ldots, \mathbb{B}_p$ turn out to be identified respectively (see the condition (c'_1) of 4.4). Here the occurrences identified with one another, regarded as symbols, will be one and the same symbols.

$(3')$ The occurrences $\mathbb{A}_1, \ldots, \mathbb{A}_p$, resp. $\mathbb{B}_1, \ldots, \mathbb{B}_p$, regarded as letters, are pairwise distinct. Consequently, the occurrences $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$, resp. $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$, regarded as letters, are pairwise distinct.

$(4')$ $\mathbb{A}'_i$ and $\mathbb{A}''_i$ are mutually variant, $i = 1, 2, \ldots, n$ (see Theorem 2.6). Likewise $\mathbb{A}'$ and $\mathbb{B}'$ are mutually variant.

$(5')$ Mutually corresponding free letters of the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ (lying in the parts $\mathbb{A}'_i$ and $\mathbb{A}''_i$) coincide (we mean occurrences of letters).

$(6')$ The bound letters of the parts $\mathbb{A}'_i$ and $\mathbb{A}''_i$ of the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ correspond to each other by groups of bound letters.

$(7')$ The substituted occurrences of the letters b'_i and b''_i in $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$ make up one or several pairs of mutually corresponding groups of bound letters (see the condition (d_{S})).

$(8')$ The groups of bound mathematical letters of the form $\mathbb{B}_{\mathsf{S}}$ of the theory $\widetilde{\mathscr{T}_{\mathsf{S}}}$ that are present in explicit form pass without change (as groups of bound letters) into the forms $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$ of the theory $\widetilde{\mathscr{T}_{\mathsf{S}}}$ (see the conditions $(a_{1\mathsf{S}})$ and (b_{S}) of 4.4 and the remark relating to the condition (b) of the definition of type I from 4.1).

$(9')$ Let $m \neq 0$, let $(n_1, \ldots, n_\ell)$ be the binding indicator of the operator sign σ, and let $m_1, \ldots, m_{n-\ell}$ be the sequence obtained from the sequence $1, 2, \ldots, n$ by deleting the members $n_1, \ldots, n_\ell$. Then for every $j \in \{1, 2, \ldots, m\}$ all the occurrences of the letters a'_j and a''_j respectively in the operators $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ and all their free occurrences respectively in the operands $\mathbb{A}'_{n_1}, \ldots, \mathbb{A}'_{n_\ell}$ and $\mathbb{A}''_{n_1}, \ldots, \mathbb{A}''_{n_\ell}$ of the indicated operators correspond to each other in the form of one pair of groups of bound letters of the parts $\mathbb{A}'$ and $\mathbb{B}'$. Consequently, the indicated occurrences of the letters a'_j and a''_j are distributed identically in the parts $\mathbb{A}'$ and $\mathbb{B}'$. To these occurrences there correspond identically distributed occurrences of the letters a'_j and a''_j in the parts $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$ of the forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ respectively. Let us denote by the symbols

$$\mathbb{A}'_{j\sigma},\ \mathbb{B}'_{j\sigma},\ \mathbb{A}'_j,\ \mathbb{B}'_j,\ \overline{\mathbb{B}}'_{\mathsf{S}j},\ \overline{\mathbb{B}}''_{\mathsf{S}j}$$

the sets of the mentioned occurrences of the letters a'_j and a''_j respectively in the operators $\sigma'\mathsf{a}'_1 \cdots \mathsf{a}'_m$ and $\sigma''\mathsf{a}''_1 \cdots \mathsf{a}''_m$ and in the parts $\mathbb{A}'$, $\mathbb{B}'$, $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$. The set $\overline{\mathbb{B}}'_{\mathsf{S}j}$, resp. $\overline{\mathbb{B}}''_{\mathsf{S}j}$, consists of all substituted occurrences of the elements of $\mathbb{A}'_{j\sigma}$, resp. $\mathbb{B}'_{j\sigma}$, and of all free occurrences of the letter a'_j, resp. a''_j, in the substituted occurrences of the parts $\mathbb{A}'_{n_1}, \ldots, \mathbb{A}'_{n_\ell}$, resp. $\mathbb{A}''_{n_1}, \ldots, \mathbb{A}''_{n_\ell}$ (see the conditions $(a_{1\mathsf{S}})$ and (d_{S}) of 4.4). From this it easily follows that the bound letters of the parts $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$ correspond to each other by groups of bound letters (see $(6')$, $(7')$, $(8')$ and the conditions $(a_{1\mathsf{S}})$, (d_{S}) and (c'_1) of 4.4).

$(10')$ In the parts $\overline{\mathbb{B}}'_{\mathsf{S}}$ and $\overline{\mathbb{B}}''_{\mathsf{S}}$, outside the substituted occurrences of the parts $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ and $\mathbb{A}''_1, \ldots, \mathbb{A}''_n$, there are no free occurrences of letters. Moreover, in the indicated substituted occurrences there occur such occurrences of the letters

$\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$ and $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$ which are respectively free occurrences in the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$. All such occurrences of any pair of letters $\overline{\mathbb{A}}_j$ and $\overline{\mathbb{B}}_j$ are identically distributed in the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$ respectively ($j = 1, 2, \ldots, p$), for they correspond to parts of the corresponding groups of bound letters of the forms $\mathbb{A}_0$ and $\mathbb{B}_0$ with canonical representatives $\mathbb{A}_j$ and $\mathbb{B}_j$ located in $\mathbb{A}'$ and $\mathbb{B}'$ (these parts are identically distributed respectively in the operands $\mathbb{A}'_1, \ldots, \mathbb{A}'_n$ and $\mathbb{A}''_1, \ldots, \mathbb{A}''_n$). Therefore the occurrences of the letters $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$ and $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$ considered respectively in the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$ make up non-empty parts of the mutually corresponding groups of bound letters of the forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ with canonical representatives $\overline{\mathbb{A}}_1, \ldots, \overline{\mathbb{A}}_p$ and $\overline{\mathbb{B}}_1, \ldots, \overline{\mathbb{B}}_p$ located in the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$ (the number of such pairs of groups is p).

Remark. If $\mathbb{A}_j$ and a'_i, regarded as symbols, coincide, then the letter (occurrence) $\mathbb{A}_j$ does not act on the parts $\mathbb{A}'_{n_1}, \ldots, \mathbb{A}'_{n_\ell}$. Consequently, in this case the letter $\mathbb{A}_j$ acts on at least one part among $\mathbb{A}'_{m_1}, \ldots, \mathbb{A}'_{m_{n-\ell}}$.

(11′) Let $\mathfrak{A}$ be a non-empty set of all free occurrences of some letter a in the part $\mathbb{A}'$ which are free occurrences in the form $\mathbb{A}_0$, and let $\mathfrak{B}$ be the set of the corresponding occurrences in the form $\mathbb{B}_0$. $\mathfrak{B}$ is the set of free occurrences of the same letter a, and it can be introduced by a complete analogue of the definition of the set $\mathfrak{A}$. Here $\mathfrak{A}$ and $\mathfrak{B}$ are respectively parts of the corresponding groups of free letters of the congruent forms $\mathbb{A}_0$ and $\mathbb{B}_0$. To the sets $\mathfrak{A}$ and $\mathfrak{B}$ there correspond the sets $\mathfrak{A}_{\mathrm{S}}$ and $\mathfrak{B}_{\mathrm{S}}$ of identically distributed free occurrences of the letter a in the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$. Here the set $\mathfrak{A}_{\mathrm{S}}$, resp. $\mathfrak{B}_{\mathrm{S}}$, can be introduced by a complete analogue of the definition of the set $\mathfrak{A}$ (in the role of $\mathbb{A}'$ and $\mathbb{A}_0$ one should take $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\mathbb{A}'_0$, resp. $\overline{\mathbb{B}}''_{\mathrm{S}}$ and $\mathbb{B}'_0$). Now it is easy to see that if some letter b has free occurrences in the form $\mathbb{A}_0$ and in its part $\mathbb{A}'$, then the groups of free occurrences of the letter b in the forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ correspond to each other.

(12′) The remaining (not considered above) groups of bound, resp. free, letters of the forms $\mathbb{A}'_0$ and $\mathbb{B}'_0$ correspond to each other (these groups lie outside the parts $\overline{\mathbb{B}}'_{\mathrm{S}}$ and $\overline{\mathbb{B}}''_{\mathrm{S}}$); moreover, the corresponding free occurrences are occurrences of one and the same letter.

From the statements given it easily follows that $\mathbb{A}'_0 \simeq \mathbb{B}'_0$.

Proof of Theorem 7.2. Denote by $\mathbb{A}_{\mathrm{S}}$ and $\mathbb{B}_{\mathrm{S}}$ the results of the substitutions indicated in the formulation of the theorem, regarded as parts of the forms $\mathbb{A}'$ and $\mathbb{B}'$. From the congruence of the forms $\mathbb{A}$ and $\mathbb{B}$ follows the mutual variance of their corresponding parts (see Theorem 2.6).

In particular, the parts $\mathbb{A}_i$ and $\mathbb{B}_i$ are mutually variant ($i = 0, 1, \ldots, m$).

Let $\mathfrak{A}'$ be any non-empty set of all occurrences of one and the same letter in the operator $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m$, and let $\mathfrak{B}'$ be the set of the corresponding occurrences of letters in the operator $\widetilde{\mathsf{S}}\,\mathsf{b}_1 \cdots \mathsf{b}_m$; then $\mathfrak{B}'$ will be the set of all occurrences of one and the same letter in the operator $\widetilde{\mathsf{S}}\,\mathsf{b}_1 \cdots \mathsf{b}_m$. It is easy to see that $\mathfrak{A}'$ and $\mathfrak{B}'$ are parts of groups of bound letters, for which the parts (possibly empty) completing them to groups of bound letters lie in the parts $\mathbb{A}_0$ and $\mathbb{B}_0$. And these last parts are made up of mutually corresponding occurrences (we mean the natural one-to-one correspondence between occurrences of symbols in the mutually congruent forms $\mathbb{A}$ and $\mathbb{B}$). From

this it follows that an element of the set $\mathfrak{A}'$, regarded as a letter, is distributed in $\mathbb{A}_0$ as free occurrences in the same way as an element of the set $\mathfrak{B}'$, regarded as a letter, is distributed in $\mathbb{B}_0$ as free occurrences. From this it follows that in the parts $\mathbb{A}_S$ and $\mathbb{B}_S$ the substituted occurrences of the parts $\mathbb{A}_i$ and $\mathbb{B}_i$ $(i = 1, 2, \ldots, m)$ are distributed *identically* (the part $\mathbb{A}_i$ may have no substituted occurrence in $\mathbb{A}_S$ even in the case when a_i has a free occurrence in $\mathbb{A}_0$; such a situation will take place exactly when a_i is a repeated member of the sequence $\mathsf{a}_1, \ldots, \mathsf{a}_m$). The parts of the word $\mathbb{A}$, located outside the part $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m\mathbb{A}_1 \cdots \mathbb{A}_m\mathbb{A}_0$ of the form $\mathbb{A}$ are "the same" as the parts of the word $\mathbb{A}'$ located outside the part $\mathbb{A}_S$ of the form $\mathbb{A}'$. Here, on the basis of the definition of substitution of the type $(—/—)$, we easily conclude that on the substituted occurrence of the part $\mathbb{A}_i$ $(i = 1, 2, \ldots, m)$ in $\mathbb{A}_S$ (if it has a substituted occurrence) only the letters of the form $\mathbb{A}'$ located outside the part $\mathbb{A}_S$ act, just as on the part $\mathbb{A}_i$ only the letters of the form $\mathbb{A}$ located outside the part $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m\mathbb{A}_1 \cdots \mathbb{A}_m\mathbb{A}_0$ act. Consequently, in the substituted occurrences of the part $\mathbb{A}_i$ in $\mathbb{A}_S$ $(i = 1, 2, \ldots, m)$ the properties of boundness and freeness of each occurrence of a letter relative to $\mathbb{A}_S$ and $\mathbb{A}'$ are the same as the properties of boundness and freeness of the corresponding occurrence of the (same) letter in $\mathbb{A}_i$ relative to the part $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m\mathbb{A}_1 \cdots \mathbb{A}_m\mathbb{A}_0$ and the form $\mathbb{A}$. Likewise, in the passage from $\mathbb{A}$ to $\mathbb{A}'$ the properties of boundness and freeness do not change for those occurrences of letters in the part $\mathbb{A}_0$ which differ from the free occurrences in $\mathbb{A}_0$ of the letters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ (although the indicated passage causes the replacement of some groups of bound letters located in $\mathbb{A}_0$, in accordance with the definition of substitution of the type $(—/—)$).

Thus, in the passage from $\mathbb{A}$ to $\mathbb{A}'$ the operator $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m$, its operands $\mathbb{A}_1, \ldots, \mathbb{A}_m$ and the free occurrences of the letters $\mathsf{a}_1, \ldots, \mathsf{a}_m$ in $\mathbb{A}_0$ disappear. Here, instead of all these free occurrences of the letter a_i there appear substituted occurrences of some part $\mathbb{A}_j$ $(j \leq i)$. Consequently, in the passage from $\mathbb{A}$ to $\mathbb{A}'$ the groups of bound letters generated by the operator $\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m$ disappear.

Complete analogues of what has been said about the form $\mathbb{A}$ and about the symbols connected with $\mathbb{A}$ are valid for the form $\mathbb{B}$ and for the analogous symbols connected with $\mathbb{B}$.

Now it is easy to see that between the occurrences of symbols in the forms $\mathbb{A}'$ and $\mathbb{B}'$ a one-to-one correspondence is naturally established: operators correspond to operators, and letters correspond to letters. Here $\mathbb{A}_S$ and $\mathbb{B}_S$ are mutually corresponding parts. In this correspondence those groups of bound letters all of whose elements are located in the parts $\mathbb{A}_S$ and $\mathbb{B}_S$ outside the substituted occurrences are mutually corresponding groups of bound letters, just as in the case of the forms $\mathbb{A}$ and $\mathbb{B}$ (the replacement of some groups of bound letters located in $\mathbb{A}_0$ does not hinder this). Above we showed that for every $i \in \{1, 2, \ldots, m\}$ the substituted occurrences of the parts $\mathbb{A}_i$ and $\mathbb{B}_i$ in the forms $\mathbb{A}_S$ and $\mathbb{B}_S$ are located identically. From this it follows that to a group of bound letters of the form $\mathbb{A}'$ located in one substituted occurrence of the part $\mathbb{A}_i$ there corresponds a group of bound letters of the form $\mathbb{B}'$ located in one substituted occurrence of the part $\mathbb{B}_i$ $(i = 1, 2, \ldots, m)$. It is easy to see that there are no other groups of bound letters located in the parts $\mathbb{A}_S$ and $\mathbb{B}_S$.

Let us consider now such mutually corresponding groups of bound letters $\mathfrak{A}$ and $\mathfrak{B}$ of the forms $\mathbb{A}$ and $\mathbb{B}$ which (as a whole) are not located in the parts $\mathbb{A}_0$ and $\mathbb{B}_0$ and do not disappear in the passage to $\mathbb{A}'$ and $\mathbb{B}'$. These are those and only those groups of bound letters whose canonical representatives are located outside the parts

$$\widetilde{\mathsf{S}}\,\mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_m \mathbb{A}_0 \qquad \text{and} \qquad \widetilde{\mathsf{S}}\,\mathsf{b}_1 \cdots \mathsf{b}_m \mathbb{B}_1 \cdots \mathbb{B}_m \mathbb{B}_0. \tag{4}$$

In the passage from the forms $\mathbb{A}$ and $\mathbb{B}$ to the forms $\mathbb{A}'$ and $\mathbb{B}'$, those parts of these groups which are located outside the parts (4) or in the parts $\mathbb{A}_0$ and $\mathbb{B}_0$ do not change (the parts located in $\mathbb{A}_0$ and in $\mathbb{B}_0$ may be empty). Denote by $\mathfrak{A}^i$ and $\mathfrak{B}^i$ the parts of the groups $\mathfrak{A}$ and $\mathfrak{B}$ located in the parts $\mathbb{A}_i$ and $\mathbb{B}_i$. Let us note that these parts of the groups $\mathfrak{A}$ and $\mathfrak{B}$ disappear, but instead, to the group $\mathfrak{A}$, for every $i \in \{1, 2, \ldots, m\}$, there are added, from every substituted occurrence of $\mathbb{A}_i$ in $\mathbb{A}_\mathsf{S}$, the occurrences corresponding to the occurrences entering into $\mathfrak{A}^i$, and to the group $\mathfrak{B}$, for every $i \in \{1, 2, \ldots, m\}$, there are added, from every substituted occurrence of $\mathbb{B}_i$ in $\mathbb{B}_\mathsf{S}$, the occurrences corresponding to the occurrences entering into $\mathfrak{B}^i$. Obviously, in this way the groups of bound letters $\mathfrak{A}$ and $\mathfrak{B}$ are transformed into groups of bound letters which correspond to each other with respect to the natural one-to-one correspondence between the occurrences in the forms $\mathbb{A}'$ and $\mathbb{B}'$.

Let us consider now mutually corresponding groups of free letters $\mathfrak{A}_i$ and $\mathfrak{B}_i$ of the forms $\mathbb{A}$ and $\mathbb{B}$. They consist of occurrences of one and the same letter. Those parts of these groups which are located outside the parts (4) or in the parts $\mathbb{A}_0$ and $\mathbb{B}_0$ do not change in the passage from the forms $\mathbb{A}$ and $\mathbb{B}$ to the forms $\mathbb{A}'$ and $\mathbb{B}'$. The parts located in the operands $\mathbb{A}_1, \ldots, \mathbb{A}_m, \mathbb{B}_1, \ldots, \mathbb{B}_m$ are transformed analogously to the preceding case, so that the groups $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are transformed into mutually corresponding groups of free letters of the forms $\mathbb{A}'$ and $\mathbb{B}'$ (the case is also meant when, of the three kinds of parts considered of the groups $\mathfrak{A}_i$ and $\mathfrak{B}_i$, one or two are empty).

Now it is obvious that $\mathbb{A}' \simeq \mathbb{B}'$. □

Since the class of contracted forms of the theory $\mathscr{T}$ coincides with the class of forms of the extended theory $\widetilde{\mathscr{T}}$, contracted forms have the properties of forms and the notions connected with forms are applied to contracted forms (without introducing their special definitions). In particular, one may speak of simplified versions of a contracted form, of parts of a contracted form, etc.

Further, since the notion of a form of the theory $\mathscr{T}$ is a particular case of the notion of a contracted form, below everywhere, where not stated otherwise, we shall assume that by the introduction of any notion relating to a contracted form the corresponding notion relating to a form of the theory $\mathscr{T}$ has been introduced. We shall use an analogue of this agreement in all cases when some general notion is considered together with its particular cases.

Now let us prove the following theorem.

Theorem 7.7. *Let the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ be obtained from the form $\mathbb{A}_0$ of the theory $\widetilde{\mathscr{T}}$ by applying to a derived operator σ its definition. Further, let the type of every derived operator occurring in at least one of the forms $\mathbb{A}_0$ and $\mathbb{A}_1$, and the type of every derived*

operator preceding any such operator, be one of I–VI, II′, IV′, VI′ (*resp.* I–VI), *and let the type of* σ *be one of* I, II, II′ (*resp.* I, II). *Then* $\mathbb{A}_0$ *and* $\mathbb{A}_1$ *are contracted versions of one and the same form* $\mathbb{A}$ *of the theory* $\mathscr{T}$ (*resp. by applying to* σ *its definition the unique value of the contracted form* $\mathbb{A}_0$ *is not changed*).

Remark. If the phrase "of the types I–VI, II′, IV′, VI′" is replaced by the phrase "of the types I–IV, VII, II′, IV′", then we obtain a proposition which will be valid, for example, in the case when among the operators of the theory $\mathscr{T}$ there occur no quantifiers with propositional operator letters (then there does not exist a derived quantifier with propositional operator letters either, and in the definitions of contracting symbols there occur no metaletters whose domains of variation contain propositional letters).

Proof. Without loss of generality one may assume that $\mathbb{A}_1$ is obtained from $\mathbb{A}_0$ by replacing the part $\sigma\,\mathbb{B}'_1\cdots\mathbb{B}'_n$ by a form $\mathbb{B}'$, where

$$\sigma\,\mathbb{B}'_1\cdots\mathbb{B}'_n \text{ ——— } \mathbb{B}' \tag{5}$$

is an instance of the definition

$$\sigma\,\mathbb{B}_1\cdots\mathbb{B}_n \text{ ——— } \mathbb{B} \tag{6}$$

of the operator σ. Obviously $\mathbb{B}$ is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables $\mathbb{B}_1,\ldots,\mathbb{B}_n,\mathsf{b}_1,\ldots,\mathsf{b}_k$,

$$\mathbb{B}' = (\mathbb{B}'_1,\ldots,\mathbb{B}'_n,\mathsf{b}'_1,\ldots,\mathsf{b}'_k\,/_:\,\mathbb{B}_1,\ldots,\mathbb{B}_n,\mathsf{b}_1,\ldots,\mathsf{b}_k)\mathbb{B},$$

$\mathbb{B}'_i$ is a form of the theory $\widetilde{\mathscr{T}}$, and $\mathsf{b}'_1,\ldots,\mathsf{b}'_k$ are letters (here n and k, independently of each other, may be equal to zero). If σ is a derived operator of type II, then the letters $\mathsf{b}'_1,\ldots,\mathsf{b}'_k$ are uniquely determined by the operator σ and the forms $\mathbb{B}'_1,\ldots,\mathbb{B}'_n$.

Denote by $\mathbb{A}^*_0$ the form of the theory $\widetilde{\mathscr{T}}$ obtained from $\mathbb{A}_0$ by transforming the parts $\mathbb{B}'_1,\ldots,\mathbb{B}'_n$ into forms $\mathbb{B}^*_1,\ldots,\mathbb{B}^*_n$ of the theory $\mathscr{T}$ by canonical applications of the definitions of derived operators, and then by applying the instance of the definition of the operator σ obtained from (6) by substituting, in place of the metavariables $\mathbb{B}_1,\ldots,\mathbb{B}_n,\mathsf{b}_1,\ldots,\mathsf{b}_k$, the system $\mathbb{B}^*_1,\ldots,\mathbb{B}^*_n,\mathsf{b}'_1,\ldots,\mathsf{b}'_k$ (here essential is the circumstance that the forms $\mathbb{B}^*_i$ and $\mathbb{B}'_i$ of the theory $\widetilde{\mathscr{T}}$ have one and the same system of free quantifier letters, and therefore, for the system $\mathbb{B}^*_1,\ldots,\mathbb{B}^*_n$ of values of the metavariables $\mathbb{B}_1,\ldots,\mathbb{B}_n$, as the system of values of the metavariables $\mathsf{b}_1,\ldots,\mathsf{b}_k$ one can take $\mathsf{b}'_1,\ldots,\mathsf{b}'_k$; resp. the system of values of the metavariables $\mathsf{b}_1,\ldots,\mathsf{b}_k$ uniquely determined by the forms $\mathbb{B}^*_1,\ldots,\mathbb{B}^*_n$ will again be $\mathsf{b}'_1,\ldots,\mathsf{b}'_k$). Further, denote by $\mathbb{A}^*_1$ the form of the theory $\widetilde{\mathscr{T}}$ obtained from $\mathbb{A}_1$ by transforming the substituted occurrences of the forms $\mathbb{B}'_1,\ldots,\mathbb{B}'_n$ in the part $\mathbb{B}'$ into forms of the theory $\mathscr{T}$ by means of the same canonical instances of the definitions of operators which we used for transforming the parts $\mathbb{B}'_1,\ldots,\mathbb{B}'_n$ of the form $\mathbb{A}_0$ into the forms[2] $\mathbb{B}^*_1,\ldots,\mathbb{B}^*_n$. It is easy to see that

[2]One may assume that in the indicated transformations the definitions were applied with the condition (d_2).

$\mathbb{A}_0^* \dashv\vdash \mathbb{A}_1^*$. From this the validity of the theorem easily follows. □

From the preceding theorem there easily follows

Theorem 7.8. *Let the type of every definition-scheme from the sequence* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of the types* I *and* II. *In order that a form* $\mathbb{A}_1$ *of the theory* $\widetilde{\mathscr{T}}$ *be a contracted version of a form* $\mathbb{A}$ *of the theory* $\mathscr{T}$, *it is necessary and sufficient that it be possible to find such a regular, resp. normal, sequence of forms of the theory* $\widetilde{\mathscr{T}}$ *for which* $\mathbb{A}_1$ *and* $\mathbb{A}$ *are end members.*

If in the formulation of this theorem we replace the phrase "one of the types I and II" by the phrase "one of the types I, II and II′", we obtain a false proposition (and the second part of this proposition will be false). Indeed, the necessity of the condition is obvious in all cases. Therefore one must give such an example of a normal sequence $\mathbb{A}_1, \ldots, \mathbb{A}_n$ of forms of the theory $\widetilde{\mathscr{T}}$ whose last member is a form of the theory $\mathscr{T}$ and $\mathbb{A}_1$ is not a contracted version of the form $\mathbb{A}_n$ (we assume that among the derived operators there occurs an operator of type II′). To obtain such an example the following suffices: as $\mathbb{A}_1$ take a form of the shape $\sigma\, \mathbb{B}_1 \cdots \mathbb{B}_m$, where $\mathbb{B}_1, \ldots, \mathbb{B}_m$ are forms of the theory $\widetilde{\mathscr{T}}$, with $\mathbb{B}_1$ containing a derived operator of type II′, and σ is such a derived operator with the definition $\sigma\, \mathbb{A}_1' \cdots \mathbb{A}_m' \text{ ——— } \mathbb{B}'$ that the metavariable $\mathbb{A}_1'$ has at least two occurrences in $\mathbb{B}'$; as $\mathbb{A}_2$ take the form of the theory $\widetilde{\mathscr{T}}$ obtained from $\mathbb{A}_1$ by applying to the operator σ its definition; further transformations are carried out in such a way that at least two parts of the form $\mathbb{A}_n$, naturally corresponding to different parts of the form $\mathbb{A}_2$, which in turn naturally correspond to the part $\mathbb{B}_1$ of the form $\mathbb{A}_1$, be different.

Nevertheless the following is valid.

Theorem 7.9. *Let the type of every definition-scheme from the sequence* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of the types* I, II *and* II′. *If* $\mathbb{A}_1, \ldots, \mathbb{A}_n$ *is a regular sequence of forms of the theory* $\widetilde{\mathscr{T}}$ *whose last member is a form of the theory* $\mathscr{T}$, *and* $\mathbb{A}_1'$ *is a form of the theory* $\mathscr{T}$ *for which* $\mathbb{A}_1$ *is a contracted version, then* $\mathbb{A}_n \simeq \mathbb{A}_1'$.

In other words: regular sequences whose first member is the given form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ and whose last member is some form of the theory $\mathscr{T}$ determine the forms of the theory $\mathscr{T}$ denoted by the form $\mathbb{A}_1$, up to congruence, in the sense that the last member of the sequence is congruent to any form of the theory $\mathscr{T}$ denoted by the form $\mathbb{A}_1$.

Proof. By virtue of Theorem 7.7 the contracted forms $\mathbb{A}_i$ and $\mathbb{A}_{i+1}$ are contracted versions of one and the same form of the theory $\mathscr{T}$ $(i = 1, 2, \ldots, n-1)$. Let $\mathbb{A}_i'$ be any such form of the theory $\mathscr{T}$ for which $\mathbb{A}_i$ is a contracted version $(i = 1, 2, \ldots, n)$. Hence, by means of the transitivity of the relation of congruence and Theorem 7.6 or 7.5, we easily conclude that

$$\mathbb{A}_1' \simeq \mathbb{A}_2' \simeq \cdots \simeq \mathbb{A}_n' \dashv\vdash \mathbb{A}_n.$$

Consequently, we have $\mathbb{A}_1' \simeq \mathbb{A}_n$. □

From Theorem 7.8 it follows that in the case when the type of every definition-scheme from the sequence $\mathrm{D}1, \mathrm{D}2, \ldots$ is one of the types I and II, the above definition of contracted versions of forms of the theory $\mathscr{T}$ is equivalent to each of the following definitions.

We shall say that a form $\mathbb{B}$ of the theory $\widetilde{\mathscr{T}}$ is a **contracted version of a form $\mathbb{A}$ of the theory $\mathscr{T}$ relative to the system of definitions** $\mathrm{D}1, \mathrm{D}2, \ldots$ if there is a normal, resp. regular, sequence of forms of the theory $\widetilde{\mathscr{T}}$ whose end members are $\mathbb{A}$ and $\mathbb{B}$. If we identify mutually congruent forms, then on the basis of the last theorem one can say that the stated assertion extends also to the case when the type of every definition-scheme from the sequence $\mathrm{D}1, \mathrm{D}2, \ldots$ is one of the types I, II and II′.

7.2 α- and α^*-Processes of Reconstructing Forms from Contracted Forms

For a given form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, the finding of such a form $\mathbb{A}$ of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version often makes use of the following so-called **α-process**, resp. **α^*-process**, of general form.

Let us choose arbitrarily a derived operator in $\mathbb{A}_1$ and apply to it its definition, resp. apply to it its definition with the condition (d_2). The resulting form $\mathbb{A}_2$ we transform in the same way into a form $\mathbb{A}_3$, and so on. We continue this process until a form of the theory $\mathscr{T}$ is obtained.

By virtue of Theorem 7.12 these processes are terminating in the cases important for practice.

Every α^*-process is an α-process. Therefore the notions relating to these processes can be introduced only for α-processes.

The transformation of the form $\mathbb{A}_i$ into the form $\mathbb{A}_{i+1}$ in the mentioned α-process we shall call the **i-th step of this α-process**. In the sequence of forms of the theory $\widetilde{\mathscr{T}}$ determined by the α-process connected with the form $\mathbb{A}_1$, the last member $\mathbb{A}$, and only it (if it exists, i.e. if the α-process is terminating), will be a form of the theory $\mathscr{T}$. Here, in the cases important for practice, $\mathbb{A}_1$ will be a contracted version of a suitably chosen form $\mathbb{A}'$ of the theory $\mathscr{T}$ which is a congruent form of the form $\mathbb{A}$, while in particular cases important for practice $\mathbb{A}_1$ will be a contracted version of the form $\mathbb{A}$.

It is easy to see that an α^*-process (and, consequently, an α-process) connected with the form $\mathbb{A}_1$ (excluding trivial cases from consideration) is not uniquely determined, and different α^*-processes connected with the form $\mathbb{A}_1$ may have different numbers of steps. Obviously an arbitrary such normal sequence of forms of the theory $\widetilde{\mathscr{T}}$ whose last member is a form of the theory $\mathscr{T}$ is determined by a uniquely determined terminating α-process and, conversely, an arbitrary terminating α-process is determined by a uniquely determined such normal sequence whose last member is a form of the theory $\mathscr{T}$.

Two α-processes α_1 and β_1 we call **corresponding** or **parallel** if the corresponding normal sequences

$$\mathbb{A}_1, \mathbb{A}_2, \ldots \qquad \text{and} \qquad \mathbb{B}_1, \mathbb{B}_2, \ldots$$

satisfy the condition: the corresponding members of these sequences are mutually congruent (and, consequently, these sequences have one and the same finite or infinite length and the corresponding members other than the first are obtained from the preceding members by applying to the corresponding operators their definitions).

On the basis of Theorem 7.4 it is easy to see that the following is valid.

Lemma 7.10. *Let the type of every member of the sequence of definition-schemes* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of* I–IV, II′ *and* IV′. *Further, let* $\mathbb{A}_1$ *and* $\mathbb{B}_1$ *be mutually congruent forms of the theory* $\widetilde{\mathscr{T}}$ *and let* α_1 *be a given* α*-process connected with* $\mathbb{A}_1$, *with the corresponding normal sequence* $\mathbb{A}_1, \mathbb{A}_2, \ldots$. *Then the* α*-process* β_1 *connected with* $\mathbb{B}_1$ *and parallel to the* α*-process* α_1, *together with the corresponding normal sequence* $\mathbb{B}_1, \mathbb{B}_2, \ldots$, *can be defined in the following way. The* i*-th step* $(i = 1, 2, \ldots)$ *of the process* β_1 *means:* (1) *to find the derived operator* σ_i' *of the form* $\mathbb{B}_i$ *corresponding to that derived operator* σ_i *of the form* $\mathbb{A}_i$ (*at the* $(i-1)$*-th step,* $i = 2, 3, \ldots$, *it is shown that* $\mathbb{A}_i \simeq \mathbb{B}_i$) *to which its definition is applied to obtain the form* $\mathbb{A}_{i+1}$; (2) *to apply to the operator* σ_i' *its definition and to note that the form* $\mathbb{B}_{i+1}$ *obtained by means of this application is congruent to the form* $\mathbb{A}_{i+1}$ (*the needed instance of the definition of the operator* σ_i' *is determined uniquely if and only if the type of the operator* σ_i' *is one of* I–IV. *In the general case one can always introduce additional restrictions which uniquely determine the needed instance of the definition of the operator* σ_i').

From the lemma follows

Theorem 7.11. *Let the type of every member of the sequence of definition-schemes* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of* I–IV, II′ *and* IV′, *and let* $\mathbb{A}_1$ *and* $\mathbb{B}_1$ *be mutually congruent forms of the theory* $\widetilde{\mathscr{T}}$. *For any* α*-process* α_1 *connected with* $\mathbb{A}_1$ *there exists a unique* α^**-process* β_1^* *connected with* $\mathbb{B}_1$ *which is parallel to the* α*-process* α_1.

For any α-process α_1 connected with some form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$, we shall denote by α_1^* the α^*-process connected with $\mathbb{A}_1$ which is parallel to the α-process α_1.

If we add propositional variables to the alphabet of the theory $\mathscr{T}$, we obtain a theory which we denote by $\mathscr{T}_{\mathrm{P}}$. If the alphabet of the theory $\mathscr{T}$ contains propositional variables, then $\mathscr{T}$ and $\mathscr{T}_{\mathrm{P}}$ coincide. In passing from $\mathscr{T}$ to the theory $\mathscr{T}_{\mathrm{P}}$ we keep the definitions introduced in $\mathscr{T}$ (including the definitions of contracting symbols). The validity of the theorem being proved for the theory $\mathscr{T}$ often easily follows from its validity in $\mathscr{T}_{\mathrm{P}}$. In such cases, if convenient, in carrying out the proof we shall assume that $\mathscr{T}$ coincides with $\mathscr{T}_{\mathrm{P}}$, without warning the reader about it (see the proof of Theorem 7.12). Analogously, from the notions connected with forms of the theory $\mathscr{T}_{\mathrm{P}}$, the corresponding notions (denoted by the same terms) for forms of the theory $\mathscr{T}$ are often naturally induced. In such cases, if convenient, in defining a notion of the indicated kind we shall assume that $\mathscr{T}$ coincides with $\mathscr{T}_{\mathrm{P}}$. We shall regard the notions thus introduced as defined also in the case when $\mathscr{T}$ does not coincide with $\mathscr{T}_{\mathrm{P}}$. Thus, for example, to define some numerical characteristic of forms of the theory $\mathscr{T}$ it suffices to formulate definitions for forms of the theory $\mathscr{T}_{\mathrm{P}}$; the indicated numerical characteristic will thereby be defined also for forms of the theory $\mathscr{T}$ (in view of the fact that every form of the theory $\mathscr{T}$ is at the same time also a form of the theory $\mathscr{T}_{\mathrm{P}}$). In such cases, without warning the reader, we shall assume that $\mathscr{T}$

coincides with $\mathscr{T}_{\mathrm{P}}$. After the formulation of the definition we shall consider that the introduced notion is defined (by the induced definition) also for such $\mathscr{T}$ that do not coincide with $\mathscr{T}_{\mathrm{P}}$.

At the end of this section and in Chapter 8 results are presented which are generalizations of results obtained in the work [4].

The following theorem we shall prove by so-called complex (nested) mathematical induction.

Theorem 7.12. *Let the type of every member of the sequence of definition-schemes* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of the types* I *and* II (*resp.* I, II *and* II′). *Then for any fixed form* $\mathbb{A}_1$ *of the theory* $\widetilde{\mathscr{T}}$ *there exists such a natural number* n_0 *that the number of steps of any* α*-process connected with* $\mathbb{A}_1$ *does not exceed* n_0. *Moreover, the last member* $\mathbb{A}_n$ *of the normal sequence of forms of the theory* $\widetilde{\mathscr{T}}$ *determined by an arbitrary* α*-process connected with* $\mathbb{A}_1$ *is a form of the theory* $\mathscr{T}$ (*resp. is congruent to such a form* $\mathbb{A}$ *of the theory* $\mathscr{T}$ *for which* $\mathbb{A}_1$ *is a contracted version*). *Besides, the last members of such sequences are graphically equal* (*resp. mutually congruent*).

Proof. It suffices to prove

Assertion (a). Let the type of every member of the sequence of definition-schemes $\mathrm{D}1, \mathrm{D}2, \ldots$ be one of I and II. Then for any fixed form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ there exists such a natural number n_0 that the number of steps of any α-process connected with $\mathbb{A}_1$ does not exceed n_0. Moreover, the last member $\mathbb{A}_n$ of the normal sequence of forms of the theory $\widetilde{\mathscr{T}}$ determined by an arbitrary α-process connected with $\mathbb{A}_1$ is determined uniquely.

Indeed, let the assertion (a) be proved. Then the first part of the theorem easily follows from the uniqueness of $\mathbb{A}_n$. The proof of the second part of the theorem is completed as follows. From the assertion (a) follows the assertion (a^*), whose formulation is obtained from the formulation of the assertion (a) if we replace in it the phrase "I and II" by the phrase "I, II and II′" and the symbol α by the symbol α^*. Consequently, one may assume that the assertion (a^*) is proved. Further, let

$$\mathbb{A}'_1 = \mathbb{A}_1, \mathbb{A}'_2, \ldots \tag{1}$$

be the normal sequence determined by an arbitrary given α-process α_1 connected with $\mathbb{A}_1$, and

$$\mathbb{A}^*_1 = \mathbb{A}_1, \mathbb{A}^*_2, \ldots \tag{2}$$

be the normal sequence determined by the α^*-process α^*_1, which is the α^*-process corresponding to the α-process α_1 (see Theorem 7.11). By virtue of the assertion (a^*) there is such a natural number n_0, depending only on $\mathbb{A}_1$, that the number m of members of the sequence (2) (and, consequently, of (1)) does not exceed n_0. Finally, from the uniqueness of the last member $\mathbb{A}^*_m$ of the sequence (2) it follows that $\mathbb{A}_1$ is a contracted version of the form $\mathbb{A}^*_m$ of the theory $\mathscr{T}$. It remains to note that $\mathbb{A}'_m \simeq \mathbb{A}^*_m$ and to apply the arbitrariness of the normal sequence (1), the uniqueness of the form $\mathbb{A}^*_m$ and the transitivity of congruence.

Let us prove the assertion (a).

Suppose that ℓ and m are positive natural numbers and that the assertion (a) is valid for all those arbitrary forms of the theory $\widetilde{\mathscr{T}}$ for which the number of occurrences of contracting symbols with number ℓ is less than m, and the number of occurrences of contracting symbols whose number is greater than ℓ is zero (in the case $\ell = 1$ and $m = 1$ the validity of the assertion (a) is proved trivially). It suffices to show the validity of the assertion for an arbitrary such form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ for which the number of occurrences of contracting symbols with number ℓ is less than $m + 1$, and the number of occurrences of contracting symbols whose number is greater than ℓ is zero.

Indeed, after carrying out the inductive step the proof of the assertion (a) is completed as follows. In the case $\ell = 1$, by virtue of (simple) mathematical induction on m the assertion (a) is valid for all forms of the theory $\widetilde{\mathscr{T}}$ the numbers of whose contracting symbols are not greater than 1. Moreover, if we suppose that the assertion (a) is valid for all forms of the theory $\widetilde{\mathscr{T}}$ the numbers of whose contracting symbols are not greater than K, then on the basis of (simple) mathematical induction on m we conclude that the assertion (a) is valid for all forms of the theory $\widetilde{\mathscr{T}}$ the numbers of whose contracting symbols are not greater than $K + 1$. Hence, with the help of (simple) mathematical induction on K, the validity of the assertion (a) follows.

Let $\mathbb{B}_1, \ldots, \mathbb{B}_K$ be such a non-empty sequence of pairwise distinct parts of the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ that for every $i \in \{1, 2, \ldots, K\}$ the principal operator σ_i of the part $\mathbb{B}_i$ is a derived operator with number ℓ and in the scope of this operator there is no other derived operator with number ℓ.[3] Obviously such parts do not overlap. Let $\mathbb{A}'_1$ be the expression for obtaining which from $\mathbb{A}_1$ it suffices to mark each contracting symbol in the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$ with a dot above and to regard them as basic symbols of the corresponding type. Then $\mathbb{A}'_1$ will be such a form of the theory $\widetilde{\mathscr{T}}$ for which the number of derived operators with number ℓ is less than m. By virtue of the assumption, the assertion (a) is valid for $\mathbb{A}'_1$.

Now let us note that the validity of the assertion (a) for the form $\mathbb{A}_1$ (and, consequently, the validity of the assertion (a)) easily follows from the following proposition.

Proposition 1. Let the type of every member of the sequence of definition-schemes $\overline{\mathrm{D}1, \mathrm{D}2, \ldots}$ be one of I and II. Further, let $\mathbb{A}_1$ be an (arbitrary) form of the theory $\widetilde{\mathscr{T}}$, and let $\mathbb{B}_1, \ldots, \mathbb{B}_K$ be pairwise non-overlapping parts of the form $\mathbb{A}_1$ for which the assertion (a) is valid. Further, let $\mathbb{A}'_1$ be the expression for obtaining which from $\mathbb{A}_1$ it suffices to mark each contracting symbol in the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$ with a dot above and to regard these marked contracting symbols as basic symbols of the corresponding type. Then $\mathbb{A}'_1$ will be a form of the theory $\widetilde{\mathscr{T}}$ (more precisely, $\mathbb{A}'_1$ will be a form of the theory $\widetilde{\mathscr{T}_1}$, where $\mathscr{T}_1$ is the extension of the theory $\mathscr{T}$ by the mentioned marked contracting symbols: a marked contracting symbol should be regarded as

[3] In this part of the proof it would suffice to consider the case $K = 1$. But we reason more generally with the aim of applying this reasoning in another part of the proof.

a completely new symbol, different from the corresponding unmarked contracting symbol). If the assertion (a) is valid for $\mathbb{A}'_1$, then the assertion (a) is valid also for the form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$.

Indeed, by applying this proposition, the proof of the assertion (a) (and, consequently, of Theorem 7.12) can be completed as follows.

By virtue of Proposition 1 it suffices to show the validity of the assertion (a) for the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$. Let us fix $i \in \{1, 2, \ldots, K\}$. The part $\mathbb{B}_i$ is representable in the form $\sigma\, \mathbb{B}_{i1} \cdots \mathbb{B}_{i\beta}$, where σ is a derived operator with number ℓ and $\mathbb{B}_{i1}, \ldots, \mathbb{B}_{i\beta}$ are its operands (the case $\beta = 0$ is not excluded). Let $\mathbb{B}'_i$ be the expression for obtaining which from $\mathbb{B}_i$ it suffices to mark each contracting symbol in the parts $\mathbb{B}_{i1}, \ldots, \mathbb{B}_{i\beta}$ with a dot above and to regard these marked contracting symbols as basic symbols of the corresponding type. Obviously, by virtue of our assumption, the assertion (a) is valid for the forms $\mathbb{B}_{i1}, \ldots, \mathbb{B}_{i\beta}$. From our assumption it also easily follows that the assertion (a) is valid for the form $\mathbb{B}'_i$ (the number of derived operators of the form $\mathbb{B}'_i$ with number ℓ may be m; namely, this will be so if and only if $m = 1$; but the first step of the α-process connected with the form $\mathbb{B}'_i$ is determined uniquely, and moreover it is a canonical application of a suitable definition not changing the value of the contracted form $\mathbb{B}'_i$, and the number of derived operators with number ℓ of the form obtained from $\mathbb{B}'_i$ by means of the mentioned first step is less than m). Hence, by virtue of Proposition 1, the validity of the assertion (a) for the part $\mathbb{B}_i$ of the form $\mathbb{A}_1$ follows.

Proof of Proposition 1. Denote by $\dot{\mathbb{B}}_1, \ldots, \dot{\mathbb{B}}_K$ the parts of the form $\mathbb{A}'_1$ naturally corresponding to the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$ of the form $\mathbb{A}_1$, and call them the **marked parts** of the form $\mathbb{A}'_1$ (of the theory $\widetilde{\mathscr{T}_1}$). The part $\dot{\mathbb{B}}_i$ of the form $\mathbb{A}'_1$ we identify with the symbol $\dot{\mathbb{B}}_i$ in such a way that instead of the part $\dot{\mathbb{B}}_i$ always, if the contrary does not follow from the context, we shall write the symbol $\dot{\mathbb{B}}_i$ (understanding it as an additional object or propositional constant, depending on whether the part $\dot{\mathbb{B}}_i$ of the form $\mathbb{A}'_1$ is a term or a formula), but in considering α-processes connected with the form $\mathbb{A}'_1$ the symbol $\dot{\mathbb{B}}_i$ (more precisely, each occurrence of the symbol $\dot{\mathbb{B}}_i$) we shall understand as the part denoted by it and, in accordance with this, we shall take into account its free letters in finding the needed instances of definitions of derived operators ($i = 1, 2, \ldots, K$).

By hypothesis, for the form $\mathbb{A}'_1$, resp. $\mathbb{B}_i$, the assertion (a) is valid. Therefore there is such a natural number m_1, resp. n'_i, that the number of steps of any α-process connected with the form $\mathbb{A}'_1$, resp. $\mathbb{B}_i$, is $\leq m_1$, resp. $\leq n'_i$. Moreover, such α-processes uniquely determine the form $\mathbb{A}'$, resp. $\mathbb{B}'_i$ ($i = 1, 2, \ldots, K$), of the theory $\mathscr{T}$. Further, let n_i be the number of occurrences of the metavariable $\dot{\mathbb{B}}_i$ in the form $\mathbb{A}'$. Obviously $n_i \geq 1$ ($i = 1, 2, \ldots, K$). Let $\mathbb{A}$ be the form of the theory $\mathscr{T}$ obtained from $\mathbb{A}'$ by (immediate) substitution of the forms $\mathbb{B}'_1, \ldots, \mathbb{B}'_K$ in place of the metavariables $\dot{\mathbb{B}}_1, \ldots, \dot{\mathbb{B}}_K$. It suffices to prove that the number of steps of α-processes connected with the form $\mathbb{A}_1$ is not greater than $m_1 + \sum_{i=1}^{K} n_i n'_i$, and that such α-processes determine a unique form of the theory $\mathscr{T}$, and that this unique form is $\mathbb{A}$.

An α-process connected with the form $\mathbb{A}_1$ we shall call **regular** if its definite ini-

tial part naturally corresponds to some α-process transforming $\mathbb{A}'_1$ into $\mathbb{A}'$ (i.e. if from the normal sequence corresponding to the last α-process, upon substitution of $\mathbb{B}_1, \ldots, \mathbb{B}_K$, parts of the form $\mathbb{A}_1$, in place of the metavariables $\dot{\mathbb{B}}_1, \ldots, \dot{\mathbb{B}}_K$, there is obtained a definite initial part of that normal sequence which is determined by the first α-process connected with the form $\mathbb{A}_1$).

It is easy to see that if an α-process connected with the form $\mathbb{A}_1$ is regular, then the number of steps of this α-process is not greater than $m_1 + \sum_{i=1}^{K} n_i n'_i$ and the last member of the corresponding normal sequence is $\mathbb{A}$.

Indeed, let $\mathbb{A}''_1$ be the last member of the normal sequence corresponding to the mentioned initial part of the given regular α-process; it will be obtained from $\mathbb{A}'$ by (immediate) substitution of the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$ of the form $\mathbb{A}_1$ in place of the metavariables $\dot{\mathbb{B}}_1, \ldots, \dot{\mathbb{B}}_K$. The remaining part of the α-process under consideration transforms independently of one another the substituted occurrences of the forms $\mathbb{B}_1, \ldots, \mathbb{B}_K$ in $\mathbb{A}''_1$. Here the number of steps used for transforming each substituted occurrence $\mathbb{B}_i$ does not exceed n'_i. For (the substituted occurrences of the forms $\mathbb{B}_1, \ldots, \mathbb{B}_K$ being pairwise non-overlapping) it makes no difference in what order the transformation of some substituted occurrence of the form $\mathbb{B}_i$ proceeds: with the help of consecutive steps from the second part of the α-process, or with the help of the same steps in the same order among themselves but distributed arbitrarily in the second part of the α-process. Moreover, obviously every substituted occurrence $\mathbb{B}_i$ is transformed by means of the second part of the α-process into $\mathbb{B}'_i$ $(i = 1, 2, \ldots, K)$ and, consequently, $\mathbb{A}''_1$ is transformed into the form $\mathbb{A}$ of the theory $\mathscr{T}$.

Therefore it suffices to prove the following proposition.

Let an α-process connected with the form $\mathbb{A}_1$ be given, and let the corresponding normal sequence of forms of the theory $\widetilde{\mathscr{T}}$ be

$$\mathbb{A}_1, \mathbb{A}_2, \ldots \tag{3}$$

Then there will be found such a regular α-process that the number of members of the corresponding normal sequence of forms is not less than the number of members of the sequence (3) (consequently, the sequence (3) is finite) and the last members of these sequences are coinciding forms of the theory $\mathscr{T}$.

For the proof of this proposition we transform the sequence (3) into a normal sequence, without decreasing the number of members and without changing the end members, whose corresponding α-process is regular.

Let σ_i be the occurrence of that derived operator to which its definition is applied in transforming the form $\mathbb{A}_i$ into the form $\mathbb{A}_{i+1}$, $i = 1, 2, \ldots$. Let us define the meaning of the term "marked occurrences of the preceding forms of the parts $\mathbb{B}_1, \ldots, \mathbb{B}_K$" in the forms of the sequence (3) (a marked occurrence of a preceding form of the part $\mathbb{B}_i$ we shall call an "i-th marked occurrence", or "i-th marked part", or a "marked part" with number i $(i = 1, 2, \ldots, K)$). In the form $\mathbb{A}_1$ the marked parts are $\mathbb{B}_1, \ldots, \mathbb{B}_K$, namely $\mathbb{B}_i$ is the i-th marked part of the form $\mathbb{A}_1$. The marked parts of the form $\mathbb{A}_1$ pairwise do not overlap and every i-th marked part is a preceding form of the form $\mathbb{B}_i$. Let $j \geq 1$ and let the i-th marked parts of the form $\mathbb{A}_j$

$(i = 1, 2, \ldots, K)$ be defined in such a way that the marked parts pairwise do not overlap and every i-th marked part is a preceding form of the form $\mathbb{B}_i$. Then the i-th marked parts of the form $\mathbb{A}_{j+1}$ are defined in the following way.

Case I. In the scope of the operator σ_j there are no marked parts and σ_j itself does not lie in a marked part. In this case $\mathbb{A}_{j+1}$ is obtained from $\mathbb{A}_j$ by replacing by a new part a part isolated from the marked parts. By definition, the i-th marked parts of the form $\mathbb{A}_{j+1}$ are those and only those parts which naturally correspond to the i-th marked parts of the form $\mathbb{A}_j$.

Case II. In the scope of the operator σ_j there are one or several marked parts of the form $\mathbb{A}_j$. Let $\mathbb{C}_1, \ldots, \mathbb{C}_{j_0}$ be the complete enumeration of such marked parts, and let $\mathbb{C}$ be the part of the form $\mathbb{A}_j$ corresponding to the operator σ_j. In this case $\mathbb{C}$ and the other (different from the parts $\mathbb{C}_1, \ldots, \mathbb{C}_{j_0}$) marked parts pairwise do not overlap. Further, let $\sigma_j\, \mathbb{E}_1 \cdots \mathbb{E}_{j_1}$ ——— $\mathbb{E}$ be the definition of the operator σ_j, and let $\sigma_j\, \mathbb{E}'_1 \cdots \mathbb{E}'_{j_1}$ ——— $\mathbb{E}'$ be that instance of this definition by applying which $\mathbb{A}_{j+1}$ is obtained from $\mathbb{A}_j$ (a fixed occurrence of the form $\sigma_j\, \mathbb{E}'_1 \cdots \mathbb{E}'_{j_1}$ in $\mathbb{A}_j$ is replaced by the form $\mathbb{E}'$), where $\mathbb{E}$ is a form of the theory $\widetilde{\mathscr{T}}$ constructed using the metavariables

$$\mathbb{E}_1, \ldots, \mathbb{E}_{j_1}, \mathsf{b}_1, \ldots, \mathsf{b}_{k_1}, \tag{4}$$

$\mathsf{b}_1, \ldots, \mathsf{b}_{k_1}$ are the additional metaletters of the definition of the derived operator σ_j, and $\mathbb{E}'$ is obtained from $\mathbb{E}$ by substituting a uniquely determined admissible system

$$\mathbb{E}'_1, \ldots, \mathbb{E}'_{j_1}, \mathsf{b}'_1, \ldots, \mathsf{b}'_{k_1}$$

of values of the metavariables (4). Then each $\mathbb{C}_i$ is a part of a suitably chosen $\mathbb{E}'_{i_0}$. Further, $\mathbb{E}_{i_0}$ has one or several occurrences in $\mathbb{E}$. To these occurrences there naturally correspond occurrences of the form $\mathbb{E}'_{i_0}$ in $\mathbb{E}'$, and to the latter occurrences there likewise naturally correspond as many occurrences of the part $\mathbb{C}_i$ in $\mathbb{E}'$. Such occurrences in $\mathbb{E}'$ ($\mathbb{E}'$ is located in $\mathbb{A}_{j+1}$ in place of the part $\sigma_j\, \mathbb{E}'_1 \cdots \mathbb{E}'_{j_1}$) are, by definition, a marked part of the form $\mathbb{A}_{j+1}$, namely a marked part with the same number as that of $\mathbb{C}_i$ $(i = 1, 2, \ldots, j_0)$. The remaining marked parts in $\mathbb{A}_{j+1}$ (they are located outside $\mathbb{E}'$) and their numbers are determined in the same way as in Case I (they correspond to those marked occurrences from $\mathbb{A}_j$ that differ from $\mathbb{C}_1, \ldots, \mathbb{C}_{j_0}$).

Case III. σ_j is an operator of a marked part $\mathbb{C}$ (σ_j may be a constant). Then the part $\mathbb{C}'$ replacing the part $\mathbb{C}$ will be a marked part of $\mathbb{A}_{j+1}$ whose number coincides with the number of $\mathbb{C}$. The remaining marked parts and their numbers are determined in the same way as in Case I (they are located outside $\mathbb{C}'$).

Obviously, in all three cases the marked parts of the form $\mathbb{A}_{j+1}$ pairwise do not overlap and every i-th marked part is a preceding form of the form $\mathbb{B}_i$.

The process described uniquely determines for every member $\mathbb{A}_j$ of the sequence (3) marked parts such that the marked parts pairwise do not overlap and every i-th marked part is a preceding form of the form $\mathbb{B}_i$.

Obviously, in a certain sense one can say that in Case I the marked parts do not change, in Case II instead of one i-th marked part several such i-th marked parts

can be generated, and only in Case III does there occur a transformation of a single marked part into a marked part with the same number (they, as forms of the theory $\widetilde{\mathscr{T}}$, differ from each other).

One may assume that the α-process corresponding to the sequence (3) is non-regular. Then, as is easily seen, there is such a $j \in \{2, 3, \dots\}$ that for σ_{j-1} Case III holds, and for σ_j Case I or Case II holds. To the operator σ_j there is naturally put in correspondence a uniquely determined occurrence of an operator in $\mathbb{A}_{j-1}$. Denote it by σ'_j. Further, denote by $\mathbb{A}_{j1}$ the form of the theory $\widetilde{\mathscr{T}}$ obtained from $\mathbb{A}_{j-1}$ by applying to σ'_j its definition.

If for σ_j Case I holds, then $\mathbb{A}_{j+1}$ is obtained from $\mathbb{A}_{j1}$ by applying to σ'_{j-1} its definition, where σ'_{j-1} is that operator of $\mathbb{A}_{j1}$ which is naturally put in correspondence with the operator σ_{j-1} (they are occurrences of one and the same operator). Thus, in this case, if in the sequence (3) we replace the member $\mathbb{A}_j$ by the form $\mathbb{A}_{j1}$, then the sequence (3) is transformed into a sequence $(3')$ for which the following condition holds.

(a) The normal sequence $(3')$ is obtained from (3) by substituting "new members" $\mathbb{A}_{j1}, \dots, \mathbb{A}_{j\gamma}$ in place of the member $\mathbb{A}_j$ in such a way[4] that "in the old members" the marked parts and their numbers do not change (we assume that for the sequence $(3')$ the above definition of marked parts and their numbers is retained). Moreover, the passage from $\mathbb{A}_{j-1}$ to $\mathbb{A}_{j1}$ goes by an operator for which Case I or Case II holds, and the passage from $\mathbb{A}_{j\gamma}$ to $\mathbb{A}_{j+1}$, as well as the passage from $\mathbb{A}_{jk}$ to $\mathbb{A}_{j,k+1}$ $(k = 1, 2, \dots, \gamma - 1)$, goes by an operator for which Case III holds.

Suppose now that for the operator σ_j Case II holds. Then to the operator σ_j there naturally corresponds an operator σ'_j of $\mathbb{A}_{j-1}$ in whose scope the marked parts lie. Denote by $\mathbb{A}_{j1}$ the form obtained from $\mathbb{A}_{j-1}$ by applying to σ'_j its definition. If σ_{j-1} lies in the scope of the operator σ'_j, then σ_{j-1} is contained in some i-th marked part $\mathbb{C}$ lying in the scope of the operator σ'_j, and to the said part $\mathbb{C}$ there correspond one or several occurrences of the form $\mathbb{C}$ in $\mathbb{A}_{j1}$. Denote these i-th marked occurrences of the form $\mathbb{A}_{j1}$ by $\mathbb{C}_1, \dots, \mathbb{C}_\gamma$. Then in these marked parts to the operator σ_{j-1} there naturally correspond the operators $\sigma^1_{j-1}, \dots, \sigma^\gamma_{j-1}$ (they are occurrences of the operator σ_{j-1}). If to these operators we apply their definitions (in any order), we obtain forms $\mathbb{A}_{j2}, \dots, \mathbb{A}_{j,\gamma+1}$, where $\mathbb{A}_{j,\gamma+1}$ coincides with the form $\mathbb{A}_{j+1}$. Now it is easy to see that if in the sequence (3) we replace the member $\mathbb{A}_j$ by the new members $\mathbb{A}_{j1}, \dots, \mathbb{A}_{j\gamma}$, then for the sequence $(3')$ obtained in this way the condition (a) holds. Finally, if σ_{j-1} does not lie in the scope of the operator σ'_j, then by substituting in the sequence (3) the form $\mathbb{A}_{j1}$ in place of $\mathbb{A}_j$ we obtain a sequence $(3')$ for which the condition (a) holds.

Let us note that in the reasoning given we rely on the fact that the parts of the form $\mathbb{A}_{j1}$ denoted by the symbols $\mathbb{C}_1, \dots, \mathbb{C}_\gamma$ are transformed identically when to the operators $\sigma^1_{j-1}, \dots, \sigma^\gamma_{j-1}$ their definitions are applied. This may fail to be so when definition-schemes of type II′ are admitted. Of special importance is also the circumstance that the admissible systems of values of additional metaletters do not change

[4] In this case $\gamma = 1$.

when the order of applying definitions of contracting symbols is changed. Finally, it is also important that the replacements corresponding to applications of definitions of contracting symbols are free.

Now suppose that the number j under consideration is chosen in the minimal way. In this case the sequence obtained from (3) in the indicated way we denote by $[2)$. The sequence obtained from $[2)$ in the same way we denote by $[3)$, and so on. Moreover, the sequence (3) we denote also by $[1)$. By a **"correct segment"** of the sequence (3) we mean its initial segment $\mathbb{A}_1, \dots, \mathbb{A}_\delta$ of maximal length satisfying the condition: every member of this segment, beginning with the second, is obtained from the preceding one by an operator for which Case I or Case II holds. In the same way the correct segment of the sequence $[i)$ $(i = 2, 3, \dots)$ is defined. Obviously $j \geq \delta + 1$ and $\delta \geq 1$. It is easy to see that the length of the correct segment of the sequence $[1 + j - \delta)$ is $\delta + 1$, and the length of the correct segment of each of the sequences $[1), [2), \dots, [j - \delta)$ is δ. The correct segment of any sequence $[i)$ represents a part of the sequence corresponding to the initial part of some suitably chosen (depending on i) regular α-process connected with the form $\mathbb{A}_1$. Therefore the length of the said correct segment does not exceed m_1. Now it is obvious that the process of obtaining from the sequence (3) the sequences $[2)$, $[3)$, etc. is terminating (otherwise the lengths of their correct segments would be unbounded from above). Let

$$\mathbb{A}_1^*, \dots, \mathbb{A}_\beta^*, \mathbb{A}_{\beta+1}^*, \dots \tag{5}$$

be the last sequence among the sequences obtained by the process considered, where β is the length of the correct segment of the sequence (5). Obviously, for every $i \in \{2, 3, \dots, \beta\}$, resp. $i \in \{\beta + 1, \beta + 2, \dots\}$, the form $\mathbb{A}_i^*$ is obtained from $\mathbb{A}_{i-1}^*$ by applying the definition of a derived operator for which Case I or Case II, resp. Case III, holds. Now, since the i-th marked parts of the form $\mathbb{A}_\beta^*$ are occurrences of the form $\mathbb{B}_i$, and for the forms $\mathbb{B}_1, \dots, \mathbb{B}_K$ the theorem is valid, the marked parts do not overlap and the operators for which Case III holds are located in the marked parts (taking into account the definition of marked parts), it follows that the sequence (5) is finite. Further, since the number of members of the sequence (5) is not less than the number of members of the sequence (3), the sequence (3) is also finite. From the finiteness of the sequence (3) it also follows that its last member is a form of the theory $\mathscr{T}$. From the construction of the sequence (5) it follows that the end members of the sequence (5) coincide with the corresponding end members of the sequence (3). Consequently, the last member $\mathbb{A}_\lambda^*$ of the sequence (5) is a form of the theory $\mathscr{T}$. Now it is easy to see that outside the marked parts of the form $\mathbb{A}_\lambda^*$ there is no derived operator (otherwise $\mathbb{A}_\lambda^*$ would not be a form of the theory $\mathscr{T}$). From this it easily follows that the α-process connected with the form $\mathbb{A}_1$ and corresponding to the sequence (5) is regular. □

From the proof of the preceding theorem there easily follows

Theorem 7.13. *Let the type of every member of the sequence of definition-schemes* $\mathrm{D}1, \mathrm{D}2, \dots$ *be one of* I, II *and* II′. *In order that the number of members of a normal sequence* $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_n$ *whose last member is a form of the theory* $\mathscr{T}$ *be minimal, resp. maximal, it*

suffices that for every $i \in \{2, 3, \ldots, n\}$ the form $\mathbb{A}_i$ be obtained from $\mathbb{A}_{i-1}$ by applying a suitable definition to such a derived operator which has a minimal, resp. maximal, scope.

Corollary. *Let the type of every member of the sequence of definition-schemes* $\mathrm{D}1, \mathrm{D}2, \ldots$ *be one of* I, II *and* II′*, and let $\mathbb{A}_1$ be a contracted version of a form $\mathbb{A}$ of the theory $\mathscr{T}$. In order that the number of members of a normal sequence of the form $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n = \mathbb{A}$ be minimal, it suffices that this sequence be canonical.*

Remark. Sometimes such definitions of contracting symbols are used to which the results obtained above cannot be extended. For example, if among the definitions of contracting symbols there occurs the definition in equation (1) of §2.5.2, then some forms of the theory $\widetilde{\mathscr{T}}$ may fail to be a contracted version of a form of the theory $\mathscr{T}$. An example of such a form is the term $\sigma\,\mathsf{x}\,\mathsf{y}$, where x and y are object variables.

Chapter 8

Some Numerical Characteristics of Derived Operators and Contracted Forms, and Algorithmic Processes of Computing Them and of Reconstructing Forms from a Contracted Form

8.1 Some Numerical Characteristics of Derived Operators and Contracted Forms and Their Basic Properties

In this chapter we shall assume that the type of every definition-scheme is one of I, II and II′, unless the contrary clearly follows from the context. In this case one can introduce some important numerical characteristics of derived operators and contracted forms.

Let $\mathbb{A}_1$ be a contracted form of the theory $\mathscr{T}$, and let $\mathbb{B}_1$ be some part of the form $\mathbb{A}_1$. Obviously, $\mathbb{A}_1$ is a form of the theory $\widetilde{\mathscr{T}}$. The maximum, resp. the minimum, of the numbers of members of normal sequences of the form

$$\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \tag{1}$$

where $\mathbb{A}_n$ is a form of the theory $\mathscr{T}$, we call the **upper**, resp. **lower**, **index of the contracted form** $\mathbb{A}_1$. Further, let a be some symbol not occurring in the alphabet of the theory $\widetilde{\mathscr{T}}$ and regarded as a (non-quantifier) propositional or object letter, depending on whether $\mathbb{B}_1$ is a formula or a term. Denote by $\mathbb{A}_1'$ the contracted form obtained from $\mathbb{A}_1$ by substituting a in place of the part $\mathbb{B}_1$. Moreover, let $\mathbb{A}$ be a form of the

theory $\mathscr{T}$ (or, more precisely, of the theory obtained from $\mathscr{T}$ by adding the symbol a as a suitable letter) for which $\mathbb{A}'_1$ is a contracted version. The number of occurrences of the letter a in $\mathbb{A}$ we call the **multiplicity of the part $\mathbb{B}_1$ of the contracted form** $\mathbb{A}_1$ (the correctness of this definition easily follows from Theorem 7.6). The same number we call the **multiplicity of the symbol (occurrence) b**, if $\mathbb{B}_1$ is the minimal part of the contracted form $\mathbb{A}_1$ containing the symbol (occurrence) b. Further, by the **multiplicity of a symbol (occurrence) b of $\mathbb{B}_1$ relative to the part $\mathbb{B}_1$** we mean the multiplicity of the symbol (occurrence) b of the form $\mathbb{B}_1$.

It is easily proved that if for the forms (1) we define "marked occurrences" in the same way as was done in the proof of Theorem 7.12 (in this case $K = 1$), then the number of marked occurrences in $\mathbb{A}_n$ is the same as the multiplicity of the part $\mathbb{B}_1$ of the form $\mathbb{A}_1$; moreover, every marked occurrence in $\mathbb{A}_n$ is congruent to any form of the theory $\mathscr{T}$ for which $\mathbb{B}_1$ is a contracted version. The last assertion holds in a strengthened form in the case when the normal sequence (1) is canonical (resp. when the type of every definition-scheme of the sequence $\mathrm{D}1, \mathrm{D}2, \ldots$ is I or II). Namely, in this case every marked occurrence in $\mathbb{A}_n$ is that form (resp. that uniquely determined form) of the theory $\mathscr{T}$ for which $\mathbb{B}_1$ is a contracted version (see Theorems 7.8 and 7.6).

Now let σ be an n-place derived operator, and let $\mathsf{c}_1, \ldots, \mathsf{c}_n$ be pairwise distinct symbols not occurring in the alphabet of the theory $\widetilde{\mathscr{T}}$, regarded as non-quantifier letters in such a way that $\sigma\,\mathsf{c}_1 \cdots \mathsf{c}_n$ is a form of the theory $\widetilde{\mathscr{T}}$ (or, more precisely, of the extended theory obtained from $\widetilde{\mathscr{T}}$ by adding the symbols $\mathsf{c}_1, \ldots, \mathsf{c}_n$ as suitable letters). The upper, resp. lower, index of the contracted form $\sigma\,\mathsf{c}_1 \cdots \mathsf{c}_n$ we call the **upper**, resp. **lower**, **index of the derived operator** σ. Further, the multiplicity of the symbol c_j of the contracted form $\sigma\,\mathsf{c}_1 \cdots \mathsf{c}_n$ we call the **j-th index** (or **index of number j) of the derived operator** σ ($j = 1, 2, \ldots, n$). In addition, by the **j-th simple index** of the derived operator σ we mean the number of occurrences of the operand c_j in that contracted form which is obtained from the contracted form $\sigma\,\mathsf{c}_1 \cdots \mathsf{c}_n$ by applying the definition of the operator σ.

The various concepts and terms, and their definitions, relating to contracted forms we shall "naturally" apply to contracted forms constructed using metavariables, where each metavariable is either a quantifier metaletter, or a non-quantifier metaletter, or a metavariable which is not a metaletter and whose domain of values is either some class of formulas or some class of terms. For this purpose the contracted forms under consideration constructed using metavariables (i.e. forms of the theory $\widetilde{\mathscr{T}}$ constructed using metavariables) we shall regard as contracted forms of the theories $\mathscr{T}_a$, where $\mathscr{T}_a$ is the $\mathscr{T}$-theory whose alphabet is obtained from the alphabet of the theory $\mathscr{T}$ by adding the metavariables occurring in the considered forms of the theory $\widetilde{\mathscr{T}}$ constructed using metavariables, in such a way that every quantifier, resp. non-quantifier, metaletter is added as an additional quantifier, resp. non-quantifier, letter of the corresponding type, and each of the remaining metavariables is added as an additional non-quantifier propositional or object letter, depending on whether its domain of values is a class of formulas or a class of terms. One should assume

that the index of an additional quantifier letter is greater than the index of any basic quantifier letter; on the basis of this convention we shall regard as defined the full analogues of the concepts and terms relating to contracted forms (in particular, to forms) in the case of contracted forms constructed using metavariables. In doing so, to simplify the arguments we shall often omit in terminology the phrase "constructed using metavariables" when there is no danger of misunderstanding. For example, instead of the expression "a form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$ constructed using metavariables is a contracted version of a form $\mathbb{B}$ of the theory $\mathscr{T}$ constructed using metavariables" we shall often use, as a simplification, the expression "$\mathbb{A}$ is a contracted version of the form $\mathbb{B}$".

In connection with the concepts introduced we note the following. Let $n \neq 0$, and let

$$\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}$$

be the definition of the contracting symbol σ, and let

$$\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m \mathbb{A}_1 \cdots \mathbb{A}_n \text{ ——— } \mathbb{B}_1$$

be the definition of the derived operator $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ (the case $m = 0$ is not excluded). Then the j-th simple index of the operator $\sigma\, \mathsf{a}'_1 \cdots \mathsf{a}'_m$ is the number of occurrences of the metavariable $\mathbb{A}_j$ in $\mathbb{B}_1$ or, what is the same, in $\mathbb{B}$. Further, let $\mathsf{b}_1, \ldots, \mathsf{b}_k$ be the additional metaletters occurring in $\mathbb{B}$ or, what is the same, in $\mathbb{B}_1$, and let

$$\mathbb{B}^* = (\mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}}}\, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B},$$

resp.

$$\mathbb{B}_1^* = (\mathsf{b}'_1, \ldots, \mathsf{b}'_k \,/_{\widetilde{\mathscr{T}}}\, \mathsf{b}_1, \ldots, \mathsf{b}_k)\mathbb{B}_1,$$

where $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ is a system of values of the metavariables $\mathsf{b}_1, \ldots, \mathsf{b}_k$ satisfying the condition: the letters $\mathsf{b}'_1, \ldots, \mathsf{b}'_k$ are distinct from one another and none of them has an occurrence in $\mathbb{B}$, resp. in $\mathbb{B}_1$, and the indices of these letters are minimal (if $k = 0$, then $\mathbb{B}^* = \mathbb{B}$ and $\mathbb{B}_1^* = \mathbb{B}_1$). By $\mathbb{B}'_1$, resp. $\mathbb{B}'$, we denote the form of the theory $\mathscr{T}$ constructed using the metavariables $\mathbb{A}_1, \ldots, \mathbb{A}_n$, resp. $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathsf{a}_1, \ldots, \mathsf{a}_m$, for which $\mathbb{B}_1^*$, resp. $\mathbb{B}^*$, is a contracted version.

Two **operators** we call **similar** if their first symbols coincide. Two contracted **forms** we call **similar** if their lengths are equal and, under the canonical one-to-one correspondence between their symbols, to every operator there corresponds a similar operator. In these definitions both basic (non-zero-place) operators and derived operators (including zero-place ones) are meant. Obviously, simple similar operators coincide.

It is easy to see that the right-hand sides of the definitions of similar derived operators are similar (in the case of simple similar derived operators they coincide); moreover, the corresponding metavariables coincide. Therefore, if $\mathbb{A}$ and $\mathbb{B}$ are similar contracted forms and $\mathbb{A}'$ and $\mathbb{B}'$ are obtained from $\mathbb{A}$ and $\mathbb{B}$ by applying their definitions to the corresponding derived operators, then $\mathbb{A}'$ and $\mathbb{B}'$ will be similar. From this it easily follows that: (1) similar contracted forms have the same upper

and lower indices (see Theorem 7.13); (2) if $\mathbb{A}$ and $\mathbb{B}$ are similar contracted forms, and $\mathbb{A}_1$ and $\mathbb{B}_1$ are their corresponding parts or symbols, then the multiplicities of these parts or symbols coincide; (3) similar derived operators have coinciding upper, resp. lower, indices; (4) similar n-place ($n > 0$) derived operators have coinciding j-th indices and coincid ing j-th simple indices ($j = 1, 2, \ldots, n$).

Hint. Introduce the concept of "parallel α-processes with respect to similarity of forms" in the same way as in 7.2. The concept of parallel α-processes with respect to congruence has already been introduced. Prove the corresponding analogues of Lemma 7.10 and Theorem 7.11 for similarity; in addition, one should prove that under such parallel α-processes the identical distribution of occurrences of the metaletter a, adopted as a non-quantifier object or propositional letter, is preserved.

Now the correctness of the following definitions is easily seen.

Let σ be a derived operator sign, and let $\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m$ be any derived operator (quantifier). Then the lower index of the operator $\sigma\, \mathsf{a}_1 \cdots \mathsf{a}_m$ we shall call the **lower index of the derived operator sign** σ. The meaning of the following terms is defined analogously: the **upper index of a derived operator sign**, the **j-th index of a derived operator sign** and the **j-th simple index of a derived operator sign**.

Between the various indices and multiplicities introduced there is a connection, which is expressed by the following two theorems.

Theorem 8.1. *Let $\mathbb{A}$ be a form of the theory $\widetilde{\mathscr{T}}$, let $\mathbb{C}$ be a part of the form $\mathbb{A}$, and let $\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_{n+1}$ be the sequence of operators (occurrences of operators) of the form $\mathbb{A}$ uniquely determined by the following conditions:* a) *the last member of the sequence is the principal operator of the part $\mathbb{C}$,*[1] *and the remaining members are derived operators;* b) *in the scope of each operator of the sequence there lie all the subsequent operators;* c) *the number of members of the sequence is maximal. For each $k \in \{1, 2, \ldots, n\}$ denote by δ_k the j-th index of the operator σ_k, where j is the natural number such that σ_{n+1} (and, consequently, $\mathbb{C}$) lies in the j-th operand of the operator σ_k. Then the multiplicity γ of the part $\mathbb{C}$ of the form $\mathbb{A}$ (or, what is the same, the multiplicity of the operator σ_{n+1}) is computed by the formula*

$$\gamma = 1\, \delta_1 \delta_2 \cdots \delta_n. \tag{2}$$

Proof. Let a be some symbol not occurring in the alphabet of the theory $\widetilde{\mathscr{T}}$ and regarded as a (non-quantifier) propositional or object letter, depending on whether $\mathbb{C}$ is a formula or a term. Denote by $\mathbb{A}'$ the contracted form obtained from $\mathbb{A}$ by substituting a in place of the part $\mathbb{C}$. The part a of the form $\mathbb{A}'$ we identify with the part $\mathbb{C}$ of the form $\mathbb{A}$. Thereby the principal operator a of the part a of the contracted form $\mathbb{A}'$ turns out to be identified with the principal operator σ_{n+1} of the part $\mathbb{C}$ of the contracted form $\mathbb{A}$. In accordance with this, below, where operators of the form $\mathbb{A}'$ are discussed, we shall assume that σ_{n+1} is the (zero-place) operator a. Denote by $\mathbb{B}_j$ that part of the contracted form $\mathbb{A}'$ whose principal operator is σ_j, $j = 1, 2, \ldots, n+1$. Obviously $\mathbb{B}_{n+1}$ is a.

[1] It may also be zero-place.

Consider the normal sequence

$$\mathbb{A}'_1 = \mathbb{A}',\ \mathbb{A}'_2,\ \ldots,\ \mathbb{A}'_m, \tag{3}$$

whose last member is a form of the theory $\mathscr{T}$ and which is uniquely determined by the following process. Let, by definition, $\mathbb{B}_j$ $(j = 1, 2, \ldots, n+1)$ be a marked occurrence of the form $\mathbb{B}_j$ in the form $\mathbb{A}' = \mathbb{A}'_1$. We shall assume that by the process introduced below, in the definition of each member of the sequence (3), "marked occurrences" and "j-th marked occurrences" $(j = 1, 2, \ldots, n+1)$ are defined as in the proof of Theorem 7.12. In this connection a j-th marked occurrence which represents an occurrence of the contracted form $\mathbb{B}_j$ we call a "marked occurrence of the contracted form $\mathbb{B}_j$", and the principal operator of any marked occurrence of the contracted form $\mathbb{B}_j$ we call a "marked occurrence of the operator σ_j" $(j = 1, 2, \ldots, n+1)$. The first member of the sequence (3) is defined by the graphic equality $\mathbb{A}'_1 = \mathbb{A}'$, and the other members are defined by applying the following general rule.

(a_i) In the last member determined so far we find the first occurrence of a derived operator that is located outside the marked occurrences of the form $\mathbb{B}_i$, and we apply to this operator its definition with condition (d_2).

Namely, first we apply the rule (a_1) to obtain the members following the member $\mathbb{A}'_1$, as many times as it can be applied. Then one should likewise apply the rule (a_2), and so on. The rule (a_{n+1}) is applied last.

By induction the following is easily proved.

The process of obtaining new members of the sequence (3) by applying the rule (a_i) terminates (see Theorem 7.12) with a form $\mathbb{A}'_{\ell_i}$ in which, outside the marked occurrences of $\mathbb{B}_i$, there are no derived operators; moreover, for every $j \in \{i, i+1, \ldots, n+1\}$ any j-th marked occurrence in $\mathbb{A}'_{\ell_i}$ is a marked occurrence of the contracted form $\mathbb{B}_j$, and for every such j the number of marked occurrences of the form $\mathbb{B}_j$ in $\mathbb{A}'_{\ell_i}$ is exactly δ_{i-1} times the number of marked occurrences of the form $\mathbb{B}_j$ in $\mathbb{A}'_{\ell_{i-1}}$ (by convention $\ell_0 = \delta_0 = 1$). Moreover, for every such j any marked occurrence of the form $\mathbb{B}_j$ in $\mathbb{A}'_{\ell_i}$ is a part of a uniquely determined marked occurrence of the form $\mathbb{B}_i$ in $\mathbb{A}'_{\ell_i}$; moreover, every marked occurrence of the form $\mathbb{B}_i$ in $\mathbb{A}'_{\ell_i}$ contains a unique marked occurrence of the form $\mathbb{B}_j$ (consequently, the number of marked occurrences of $\mathbb{B}_j$ in $\mathbb{A}'_{\ell_i}$ is the same as the number of marked occurrences of the form $\mathbb{B}_i$ in $\mathbb{A}'_{\ell_i}$).

Let us note that from the maximality property of the sequence $\sigma_1, \ldots, \sigma_{n+1}$ (see c)) there follows the validity of the following proposition. The marked occurrences do not lie in the scope of those derived operators to which their definitions are applied in order to obtain new members of the sequence by the rule (a_1). Therefore, just as in $\mathbb{A}'_1$, in every member obtained by applying the rule (a_1) each $\mathbb{B}_j$ $(j = 1, 2, \ldots, n+1)$ has a unique marked occurrence.

Now it is easy to see that (3) is a normal sequence, that its last member $\mathbb{A}'_m$ is a form of the theory $\mathscr{T}$, and that in this member the number of occurrences of the form $\mathbb{B}_{n+1}$ (i.e. of the form a) is $\delta_0\delta_1 \cdots \delta_n$. Consequently, since by Theorem 7.9 the form $\mathbb{A}'_m$ is congruent to every form of the theory $\mathscr{T}$ (extended by the letter a) for

which $\mathbb{A}'_1$ is a contracted version, we have $\gamma = 1\,\delta_1\delta_2\cdots\delta_n$. □

Theorem 8.2. *Let $\mathbb{A}$ be a form of the theory $\widetilde{\mathscr{T}}$, and let $\sigma_1, \ldots, \sigma_n$ be an enumeration of all (occurrences of) derived operators located in $\mathbb{A}$. Denote by γ_i the multiplicity of the operator σ_i of the form $\mathbb{A}$, and by α_i and β_i, respectively, the lower and the upper index of the operator σ_i, $i = 1, 2, \ldots, n$. Then the lower index $\underline{\mathrm{J}}(\mathbb{A})$ and the upper index $\bar{\mathrm{J}}(\mathbb{A})$ of the form $\mathbb{A}$ are computed by the following equalities:*

$$\underline{\mathrm{J}}(\mathbb{A}) = 1 + \sum_{i=1}^{n}(\alpha_i - 1), \tag{4}$$

$$\bar{\mathrm{J}}(\mathbb{A}) = 1 + \sum_{i=1}^{n}\gamma_i(\beta_i - 1). \tag{5}$$

Proof. First we prove equality (4). In the case $n = 0$ the validity of (4) is obvious. Assume that $n > 0$.

A normal sequence of the form

$$\mathbb{A}_1 \eqcirc \mathbb{A}, \mathbb{A}_2, \ldots, \mathbb{A}_\ell \eqcirc \mathbb{A}^*, \tag{6}$$

whose last member is a form of the theory $\mathscr{T}$, with a minimal number of members, can be constructed by the following process. The first member $\mathbb{A}_1$ is defined by the graphic equality $\mathbb{A}_1 \eqcirc \mathbb{A}$. Let i_1 be such a natural number from $\{1, 2, \ldots, n\}$ that σ_{i_1} is the operator of the form $\mathbb{A}_1$ with a minimal scope and located to the left of all such operators. Denote by $\mathbb{B}_1$ the part of the form $\mathbb{A}_1$ with the principal operator σ_{i_1}. Further, let

$$\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_{j_1} \eqcirc \mathbb{B}^* \tag{7}$$

be the (canonical) normal sequence whose last member is a form of the theory $\mathscr{T}$, uniquely determined by the condition: for every $i \in \{2, 3, \ldots, j_1\}$ the form $\mathbb{B}_i$ is obtained from $\mathbb{B}_{i-1}$ by applying a suitable definition with condition (d_2) to the derived operator with minimal scope that is located to the left of the other such operators. By Theorem 7.13, the number of members of the sequence (7) is minimal among the numbers of members of normal sequences for which $\mathbb{B}_1$ is the first member and the last member is a form of the theory $\mathscr{T}$. It is easily seen that $j_1 = \alpha_{i_1}$ (this follows easily from the natural one-to-one correspondence between the set of normal sequences connected with the forms $\mathbb{B}_1$ and $\mathbb{B}_0$, where $\mathbb{B}_0$ is the form obtained from $\mathbb{B}_1$ by simultaneously replacing all operands of the operator σ_{i_1} by mutually distinct new symbols, each of which is suitably regarded as a non-quantifier propositional or object letter). Now the $\alpha_{i_1} - 1$ members of the sequence (6) following the first one are defined as follows: for every $i \in \{2, 3, \ldots, \alpha_{i_1}\}$ the form $\mathbb{A}_i$ is obtained from $\mathbb{A}_1$ by substituting the form $\mathbb{B}_i$ in place of the part $\mathbb{B}_1$. From the last of the members of the sequence constructed in this way, new members are obtained in the same way. And so on; this process terminates with the construction of the normal sequence we need (see Theorem 7.13). Now the validity of equality (4) is obvious.

Next we prove equality (5) by induction on n. The validity of (5) in the case

$n = 0$ is obvious. Therefore it suffices to prove (5) assuming that $n > 0$ and that the analogue of equality (5) is valid for all forms of the theory $\widetilde{\mathscr{T}}$ the number of derived operators of which is less than n.

Choose a natural number $\ell \in \{1, 2, \ldots, n\}$ satisfying the condition: in the scope of the operator σ_ℓ there are no derived operators. Denote by $\mathbb{C}$ the part of the contracted form $\mathbb{A}$ with the principal operator σ_ℓ. Further, denote by $\mathbb{A}'$ the form obtained from $\mathbb{A}$ by substituting in place of the part $\mathbb{C}$ a symbol a not occurring in the alphabet of the theory $\widetilde{\mathscr{T}}$ and regarded as a non-quantifier propositional or object letter, depending on whether $\mathbb{C}$ is a formula or a term.

Further, let

$$\mathbb{A}'_1 = \mathbb{A}', \mathbb{A}'_2, \ldots, \mathbb{A}'_m \tag{8}$$

be such a normal sequence that its last member is a form of the theory $\mathscr{T}$ and each member, beginning with the second, is obtained from the preceding one by applying a suitable definition with condition (d_2) to the operator with maximal scope.

By Theorem 7.13, from the inductive hypothesis it follows that

$$m = \bar{\mathsf{J}}(\mathbb{A}') = 1 + \sum_{i=1}^{n} \gamma_i(\beta_i - 1) - \gamma_\ell(\beta_\ell - 1). \tag{9}$$

It is easily seen that

$$\bar{\mathsf{J}}(\mathbb{C}) = \beta_\ell$$

(it is understood that the upper index of any contracted form $\mathbb{D}$ is denoted by $\bar{\mathsf{J}}(\mathbb{D})$). Let

$$\mathbb{A}_1 = \mathbb{A}, \mathbb{A}_2, \ldots, \mathbb{A}_m, \mathbb{A}^1_1, \ldots, \mathbb{A}^1_K \tag{10}$$

be the sequence of forms uniquely determined by the following conditions: 1) $\mathbb{A}_i$ $(i = 1, 2, \ldots, m)$ is obtained from $\mathbb{A}'_i$ by substituting $\mathbb{C}$ in place of a; 2) for each $i \in \{1, 2, \ldots, K\}$ the contracted form $\mathbb{A}^1_i$ is obtained from the preceding member by applying a suitable definition with condition (d_2) to the derived operator with maximal scope that is located to the left of all other such operators; 3) $\mathbb{A}^1_K$ is a form of the theory $\mathscr{T}$. Obviously (10) is a normal sequence. On the basis of Theorem 7.13 it is easily seen that

$$m + K = \bar{\mathsf{J}}(\mathbb{A}). \tag{11}$$

Further, let

$$\mathbb{C}_1 = \mathbb{C}, \mathbb{C}_2, \ldots, \mathbb{C}_\tau = \mathbb{C}^*$$

be the sequence of forms of the theory $\widetilde{\mathscr{T}}$ uniquely determined by the conditions: 1) for each $i \in \{2, 3, \ldots, \tau\}$ the form $\mathbb{C}_i$ is obtained from the preceding member by applying a suitable definition with condition (d_2) to the derived operator with maximal scope that is located to the left of all other such operators; 2) $\mathbb{C}_\tau$ is a form of the theory $\mathscr{T}$.

It is easily seen that the process of obtaining the members $\mathbb{A}^1_1, \ldots, \mathbb{A}^1_K$ of the sequence (10) first transforms the first occurrence of $\mathbb{C}$ in the form $\mathbb{A}_m$ into the form

$\mathbb{C}^*$, and $\beta_\ell - 1$ steps are spent on this; then this process transforms into $\mathbb{C}^*$ the first occurrence of the form $\mathbb{C}$ in the form $\mathbb{A}^1_{\beta_\ell - 1}$ (i.e. the occurrence which is naturally put in correspondence with the second occurrence of the form $\mathbb{C}$ in $\mathbb{A}_m$), and so on. Then it is obvious that $K = \gamma(\beta_\ell - 1)$, where γ is the number of occurrences of the letter a in $\mathbb{A}'_m$; i.e. γ is the multiplicity of the part $\mathbb{C}$ of the contracted form $\mathbb{A}$ or, what is the same, the multiplicity of the operator σ_ℓ of the form $\mathbb{A}$; i.e. $\gamma = \gamma_\ell$. Consequently, $K = \gamma_\ell(\beta_\ell - 1)$. Therefore (5) follows from (9) and (11). □

8.2 Algorithms for Reconstructing a Form from a Contracted Form and for Computing Numerical Characteristics of Derived Operators and Contracted Forms

The computation of any index of a derived quantifier $\sigma'\mathsf{a}_1 \cdots \mathsf{a}_m$ is equivalent to the computation of the corresponding index of the operator sign σ' (they are equal to each other).

The computation of the j-th simple index of a derived operator (or of a derived operator sign) presents no difficulty, whereas the j-th index of a derived operator σ is computed by the following algorithm.

1. Make a list of all those *simple* derived operators and derived operator signs on which the derived operator σ depends. Divide these operator signs and operators, including σ, into groups: to the i-th group we assign the operators and operator signs with number i (σ may also be a quantifier). Let $\sigma^i_1, \ldots, \sigma^i_{n_i}$ be an enumeration of the derived operators and derived operator signs of the i-th group ($i = 1, 2, \ldots, \ell$). Some of these groups may be empty. We shall assume that the last group is nonempty. Obviously it contains the unique operator σ. Denote by $n(i, k)$ the weight of the operator σ^i_k, or the first index of the weight of the derived operator sign σ^i_k ($i = 1, 2, \ldots, \ell$; $k = 1, 2, \ldots, n_i$).

2. For each $k \in \{1, 2, \ldots, n_1\}$ compute the j-th index of the derived operator or derived operator sign σ^1_k ($j = 1, 2, \ldots, n(1, k)$). Here, in order to compute the j-th index of the operator or operator sign σ^1_k, it suffices to write out the definition of any derived operator of the form $\sigma^1_k\mathsf{a}_1 \cdots \mathsf{a}_\gamma$ (if σ^1_k is a derived operator, then $\gamma = 0$) and to count in the right-hand side of this definition the occurrences of the j-th operand (metavariable) of the operator $\sigma^1_k\mathsf{a}_1 \cdots \mathsf{a}_\gamma$.

3. For each $k \in \{1, 2, \ldots, n_2\}$ compute the j-th index of the derived operator or derived operator sign σ^2_k ($j = 1, 2, \ldots, n(2, k)$), and so on. Here these computations proceed according to the following general (recursive) rule.

(1) Let $1 < m \leq \ell$, and suppose that for every non-zero natural number $d < m$ the indices of all numbers of each operator or operator sign of the enumeration $\sigma^d_1, \ldots, \sigma^d_{n_d}$ have been computed. Then for each $k \in \{1, 2, \ldots, n_m\}$ the j-th index of the derived operator or derived operator sign σ^m_k ($j = 1, 2, \ldots, n(m, k)$) is computed as follows.

Write out the definition of any derived operator of the form $\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma$ (if σ_k^m is a derived operator, then $\gamma = 0$):

$$\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma \mathbb{A}_1 \cdots \mathbb{A}_{n(m,k)} \text{ ——— } \mathbb{B}.$$

Let $\mathbb{A}_{j1}, \ldots, \mathbb{A}_{j\beta}$ be the enumeration of the occurrences of the metavariable $\mathbb{A}_j$ in $\mathbb{B}$ (j is assumed to be fixed). For each $i \in \{1, 2, \ldots, \beta\}$ compute the multiplicity γ_i of the part $\mathbb{A}_{ji}$ of the contracted form $\mathbb{B}$ (it is computed by means of equality (2) of Theorem 8.1—by the inductive hypothesis all the numbers occurring in the right-hand side of equality (2) of Theorem 8.1 have been computed). Then the j-th index of the operator or operator sign σ_k^m will be $\sum_{i=1}^{\beta} \gamma_i$.

The lower, resp. upper, index of a derived operator σ is computed by the following algorithm.

1. Carry out all the instructions of the preceding algorithm (computing the indices of all numbers for each derived operator or derived operator sign σ_k^i).

2. For each $k \in \{1, 2, \ldots, n_1\}$ both the lower and the upper index of the derived operator or derived operator sign σ_k^1 is 2.

3. For each $k \in \{1, 2, \ldots, n_2\}$ compute both the lower and the upper index of the derived operator or derived operator sign σ_k^2, and so on (this process terminates with the computation of the lower and upper indices of the operator σ). Here the computations proceed according to the following general (recursive) rule.

(1) Let $1 < m \leq \ell$, and suppose that for every non-zero natural number $d < m$ both the lower and the upper indices of the operators or operator signs $\sigma_1^d, \ldots, \sigma_{n_d}^d$ have been computed. Then for each $k \in \{1, 2, \ldots, n_m\}$ both the lower and the upper index of the operator or operator sign σ_k^m are computed as follows. Let $\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma$ be any derived operator (if σ_k^m is a derived operator, then $\gamma = 0$). Further, let $\mathsf{c}_1, \ldots, \mathsf{c}_{n(m,k)}$ be pairwise distinct symbols not occurring in the alphabet of the theory $\widetilde{\mathscr{T}}$, regarded as non-quantifier letters in such a way that $\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma \mathsf{c}_1 \cdots \mathsf{c}_{n(m,k)}$ is a form of the theory $\widetilde{\mathscr{T}}$. To the contracted form $\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma \mathsf{c}_1 \cdots \mathsf{c}_{n(m,k)}$ we shall apply the definition of the derived operator $\sigma_k^m \mathsf{a}_1 \cdots \mathsf{a}_\gamma$, and obtain the contracted form $\mathbb{B}$. Then the lower, resp. upper, index of the derived operator or derived operator sign σ_k^m will be the lower, resp. upper, index of the form $\mathbb{B}$, increased by one. These lower and upper indices $\underline{\mathsf{J}}(\mathbb{B})$ and $\bar{\mathsf{J}}(\mathbb{B})$ of the form $\mathbb{B}$ are computed by means of equalities 8.1 (4) and 8.1 (5) of Theorem 8.2 (the numbers occurring in the right-hand sides of these equalities, denoted by the symbols α_i and β_i, have been computed by the inductive hypothesis, i.e. have been computed in the course of applying this algorithm up to the present stage, and the numbers denoted by the symbols γ_i are easily computed on the basis of Theorem 8.1 and the numbers computed under item 1.

Having an algorithm for computing arbitrary indices of a derived operator, Theorem 8.1 gives an algorithm for computing the multiplicity of an arbitrary part (of an arbitrary operator) of a given contracted form. Namely, one should write out equality (2) of Theorem 8.1 and compute the numbers occurring in the right-hand side of the equality by means of the first algorithm given above. Likewise, having an

algorithm for computing the lower and upper indices of derived operators and an algorithm for computing the multiplicity of any part of a form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$, the equalities (4) and (5) of Theorem 8.2 give, respectively, algorithms for computing the lower and upper indices of an arbitrary form $\mathbb{A}$ of the theory $\widetilde{\mathscr{T}}$.

Now it is easy to see that for any form $\mathbb{A}_1$ of the theory $\widetilde{\mathscr{T}}$ the search for such a form $\mathbb{A}$ of the theory $\mathscr{T}$ for which $\mathbb{A}_1$ is a contracted version can be carried out by an algorithm obtained by any such modification of the general α-process that guarantees the uniqueness of the process (the number of steps of such a process cannot be greater than the upper index of the form $\mathbb{A}_1$). Thus, for example, the uniqueness of the said process is guaranteed by each of the following additional requirements: at each step the definition is applied with condition (d_2) to the operator with minimal, resp. maximal, scope that stands to the left (right) of all other such operators, and so on.

Remark. On the basis of the results obtained in this work one can draw the following conclusion. In expounding any classical formal or informal (content) mathematical theory one should use contracting symbols of types I–IV, II′ and IV′. In that case one can apply the results obtained above when operating on contracted forms. Instead of contracted forms one can also use their "simplified versions". The symbols used for simplifying contracted forms should be called "simplifying symbols". Having no general laws for operating directly on simplified versions of contracted forms, one should represent them to oneself as the corresponding contracted forms.

Index of Terms and Notation

Editorial note on the index. The terms are rearranged in English alphabetical order, followed by the notation in its order of occurrence in the text. All locators are *printed pages of the Russian original*. Each locator links to the beginning of the corresponding original page in this translation. A range refers to the complete original-page range, not to the page numbers of the English PDF. This avoids attributing material to an extra English page merely because the following original-page marker falls on it. Omitted process entries have been restored; the corrected locators and terminology are recorded in the revision log.

In the index, the terms are given first, and then the notation. An asterisk "*" at the end of a term marks a term that is defined implicitly in the text (in the original the pages are sometimes also indicated where the agreements implicitly defining the given term are stated). Some terms do not occur in the index in explicit form, but it is assumed that they are present implicitly. An example of such a term: "n-place special relational operator"; the sense of this term is assumed to be naturally induced from the content of the terms "n-place operator", "special operator" and "relational operator". A double asterisk "**" at the end of a term indicates that its sense should be understood intuitively, without a precise definition; an example of such a term is "simplified version of a form". An asterisk before a term means that the sense of the term is regarded as known.

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Georgian Summary and English Translation

Editorial note. The Georgian summary on printed pp. 194–195 is transcribed below from the scan of the typescript, followed by its English translation. The author's original English summary is retained separately after it.

შ. ფხაკაძე

აღნიშვნების თეორიის ზოგიერთი საკითხი

რეზიუმე

მოძებნილია საკმარისად ზოგადი სისტემა შემამოკლებელ სიმბოლოთა განსაზღვრების წესებისა კლასიკურ ფორმალურ და შინაარსულ მათემატიკურ თეორიების შემთხვევაში. იგი ოპტიმალურია იმ აზრით, რომ, ერთი მხრივ, ამ წესების გამოყენებით შესაძლებელია თითქმის ყველა იმ შემამოკლებელ სიმბოლოს შემოტანა, რომლებიც გამოიყენება ძირითად კლასიკურ ფორმალურ და შინაარსულ მათემატიკურ თეორიებში. მეორე მხრივ კი, ამ წესებით შემოტანილ შემამოკლებელ სიმბოლოთა ნებისმიერ სისტემას აქვს ისეთი თვისებები, რომლებიც უზრუნველყოფენ მნიშვნელოვან თავისუფლებას შემოკლებულ ფორმებზე ოპერირებისას.

შესწავლილია შემოკლებულ ფორმიდან ფორმის აღდგენის ალგორითმული პროცესები. საკმაოდ ბუნებრივად შემოტანილ ალგორითმთა კლასში მოძებნილია შემოკლებულ ფორმიდან ფორმის აღდგენის ალგორითმი ბიჯების მინიმალური /მაქსიმალური/ რიცხვით. შეისწავლება აგრეთვე შემამოკლებელ სიმბოლოებისა და შემოკლებულ ფორმათა სხვადასხვა რიცხვითი მახასიათებლები.

English translation.

Sh. Pkhakadze. Some Questions of Notation Theory.

A sufficiently general system of rules for defining contracting symbols in classical formal and informal mathematical theories has been found. It is optimal in the following sense. On the one hand, these rules allow the introduction of almost all contracting symbols used in the basic classical formal and informal mathematical theories. On the other hand, every system of contracting symbols introduced by these rules has properties that provide considerable freedom in operating on contracted forms.

The algorithmic processes for reconstructing a form from a contracted form are studied. Within a fairly naturally introduced class of algorithms, an algorithm for

reconstructing a form from a contracted form with a minimal (maximal) number of steps has been found. Various numerical characteristics of contracting symbols and contracted forms are also studied.

Author's English Summary

The following is the author's original English summary, reproduced with obvious typographical errors corrected. A translation of the Georgian summary is given separately above.

Sh. Pkhakadze. Some Problems of the Notation Theory.

A sufficiently general system of rules of definition of contracting symbols is found in the case of classical formal mathematical theories. The system is optimal in the sense that, by using the above rules, one can, on the one hand, introduce nearly all the contracting symbols which are used in basic classical formal and informal mathematical theories. On the other hand, any system of contracting symbols introduced according to these rules has such properties which allow one to operate freely on contracted forms.

The algorithmic processes of regenerating the form from the contracted form are studied. In the class of algorithms of regenerating the form from the contracted form, the algorithm with the minimal (maximal) number of steps is introduced. Some numerical characteristics of contracting symbols and contracted forms are also studied.

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Afterword by the Editor-translators of the First English Pre-Editorial Edition

Aleksandre Maskharashvili, Davit Gegia, and Konstantine Pkhakadze, editor-translators of the first English edition of Shalva Pkhakadze's monograph *Some Questions of Notation Theory*

This edition was prepared at the Research Institute for the Cultural Protection and Technological Development of the State Languages of Georgia, at Georgian Technical University. The Institute was founded on 15 May 2025 with the support of the university's rector, Academician Davit Gurgenidze, and is funded directly by the Ministry of Education, Science and Youth of Georgia from the state budget.

The edition was prepared as part of the founding project of the Institute, entitled "Towards Europe with a Georgia peacefully united through the Georgian and Abkhazian languages, cultures and traditions; in brief, into the European Union with the Georgian and Abkhazian languages; or, the 2025–2029 budgetary action plan for comprehensive technological support for the state languages of Georgia" (https://geoanbani.com/Other/PDFs/Articles/26.pdf).

The editor-translators:

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We welcome reports of errors of any kind, including errors in spelling, grammar, syntax,

or meaning. Please send corrections to gllc.ge@gmail.com. Your comments will help us improve future editions.

A brief biographical note follows on Professor Shalva Pkhakadze, Honored Scientist of Georgia and founder of the Georgian school of classical mathematical logic:

1. http://www.nplg.gov.ge/bios/en/00009469/ – Biographical Dictionary – Shalva Pkhakadze
2. https://gmu.gtu.ge/index.php?page=pkhakadze&lang=en – Shalva Pkhakadze 100 – Georgian Mathematical Union
3. https://newspaper.tsu.ge/uploads/newspaper/N-7%20(2233)-2019_web.pdf#page=13 – Shalva Phkhakadze 100 – At the Origins of the Ilia Vekua Institute of Applied Mathematics
4. https://geoanbani.com/TbiLLAI/ – First Tbilisi International Summer School in "Logic, Language, Artificial Intelligence" dedicated to the memory of the founder of the school of classical mathematical logic in Georgia, the honor scientist of Georgia, Shalva Pkhakadze
5. https://geoanbani.com/TbiLLAI/files/%e1%83%9e%e1%83%a0%e1%83%9d%e1%83%92%e1%83%a0%e1%83%90%e1%83%9b%e1%83%90.pdf#page=30 – The founder of the school of classical mathematical logic in Georgia, the honor scientist of Georgia, Shalva Pkhakadze – Brief Biographical Data
6. https://geoanbani.com/TbiLLAI/files/%e1%83%9e%e1%83%a0%e1%83%9d%e1%83%92%e1%83%a0%e1%83%90%e1%83%9b%e1%83%90.pdf – FIRST TBILISI INTERNATIONAL SUMMER SCHOOL IN "LOGIC, LANGUAGE, ARTIFICIAL INTELLIGENCE" dedicated to the memory of the founder of the school of classical mathematical logic in Georgia, the honor scientist of Georgia, Shalva Pkhakadze
7. https://geoanbani.com/TbiLLAI/files/AMIM%20Vol.25%20No.2,%202020.pdf – AMIM, 25, No. 2, 2020, ISSN 1512-0074 (https://www.viam.science.tsu.ge/Ami/2020_2/2020_2.htm)
8. https://geoanbani.com/TbiLLAI/files/AMIM%20Vol.25%20No.2,%202020.pdf